OLYMPIAD CORNER

No. 369

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 15, 2019**.

OC411. Show that for all integers $k > 1$ there is a positive integer m less than k^2 such that $2^m - m$ is divisible by k.

OC412. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y

$$f(y - xy) = f(x)y + (x - 1)^2 f(y).$$

OC413. To each sequence consisting of n zeros and n ones is assigned a number which is the number of largest segments with the same digits in it (for example, the sequence 00111001 has 4 such segments 00, 111, 00, 1). For each n, add all the numbers assigned to each sequence. Prove that the resulting sum is equal to

$$(n + 1)\binom{2n}{n}.$$

OC414. Find all prime numbers p and all positive integers a and m such that $a \leq 5p^2$ and $(p - 1)! + a = p^m$.

OC415. Let n be a positive integer and let $f(x)$ be a polynomial of degree n with real coefficients and n distinct positive real roots. Is it possible for some natural integer $k \geq 2$ and for a real number a that the polynomial

$$x(x + 1)(x + 2)(x + 4)f(x) + a$$

is the k-th power of a polynomial with real coefficients?

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC411. Démontrer que pour tout entier $k > 1$ il existe un entier positif m, plus petit que k^2, tel que $2^m - m$ est divisible par k.

OC412. Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que

$$f(y - xy) = f(x)y + (x-1)^2 f(y)$$

pour tous nombres réels x et y.

OC413. À chaque suite comprenant n zéros et n uns on associe un nombre égal au nombre de segments utilisant un seul chiffre (par exemple, la suite 00111001 possde 4 tels segments: 00, 111, 00, 1). Pour n donné, soit la somme des nombres associés à toutes ses suites. Démontrer que cette somme est égale à

$$(n+1)\binom{2n}{n}.$$

OC414. Déterminer tous les nombres premiers p et tous les entiers a et m tels que $a \leq 5p^2$ et $(p-1)! + a = p^m$.

OC415. Soit n un entier positif et soit $f(x)$ un polynôme de degré n à coefficients réels, ayant n racines réelles distinctes et positives. Est-ce possible que pour un certain entier $k \geq 2$ et pour un certain a réel le polynôme

$$x(x+1)(x+2)(x+4)f(x) + a$$

soit la k-ième puissance dun polynôme à coefficients réels?

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(1), p. 13–15.

OC351. Solve the system of equations

$$\begin{aligned} 6x - y + z^2 &= 3, \\ x^2 - y^2 - 2z &= -1, \\ 6x^2 - 3y^2 - y - 2z^2 &= 0. \end{aligned}$$

where $x, y, z \in \mathbb{R}$.

Originally Problem 1 of Day 1 of 2016 Vietnam National Olympiad.

We received 7 submissions of which 6 were correct and complete. We present 2 solutions.

Solution 1, by Dan Daniel.

First equation gives $y = z^2 + 6x - 3$; second equation gives $y^2 = x^2 - 2z + 1$. Substituting these values into the third equation, we get

$$6x^2 - 3(x^2 - 2z + 1) - (z^2 + 6x - 3) - 2z^2 = 0 \implies 3x^2 - 3z^2 - 6x + 6z = 0,$$

i.e.

$$(x - z)(3(x + z) - 6) = 0 \implies (x - z)(x + z - 2) = 0.$$

We have two cases.

(i) $x = 2 - z$. Then, the first equation of the system becomes

$$6(2 - z) - y + z^3 = 3 \implies y = (z - 3)^2$$

and the second equation of the system becomes

$$(2 - z)^2 - (z - 3)^4 - 2z + 1 = 0 \implies (z - 3)^4 - (z - 3)^2 + 4 = 0.$$

Setting $t = z - 3$, we have no solutions in this case.

(ii) $z = x$. Then, $y^2 = x^2 - 2x + 1$, i.e. $y = \pm(x - 1)$. If $y = x - 1$, then from the first equation of the system, we get

$$x^2 + 5x - 2 = 0 \implies (x, y, z) = \left(\frac{-5 \pm \sqrt{33}}{2}, \frac{-7 \pm \sqrt{33}}{2}, \frac{-5 \pm \sqrt{33}}{2}\right).$$

If $y = 1 - x$, then from the first equation of the system, we get

$$x^2 + 7x - 4 = 0 \implies (x, y, z) = \left(\frac{-7 \pm \sqrt{65}}{2}, \frac{9 \mp \sqrt{65}}{2}, \frac{-7 \pm \sqrt{65}}{2}\right).$$

Solution 2, by Oliver Geupel.

It is tedious but straightforward to check that the following four triplets (x, y, z) are solutions to the problem:

$$\left(\frac{-7+\sqrt{65}}{2}, \frac{9-\sqrt{65}}{2}, \frac{-7+\sqrt{65}}{2}\right), \left(\frac{-7-\sqrt{65}}{2}, \frac{9+\sqrt{65}}{2}, \frac{-7-\sqrt{65}}{2}\right),$$

$$\left(\frac{-5+\sqrt{33}}{2}, \frac{-7+\sqrt{33}}{2}, \frac{-5+\sqrt{33}}{2}\right), \left(\frac{-5-\sqrt{33}}{2}, \frac{-7-\sqrt{33}}{2}, \frac{-5-\sqrt{33}}{2}\right).$$

We show that there is no further solution.

Suppose (x, y, z) is any solution. Then,

$$0 = (3-6x+y-z^2)-3(x^2-y^2-2z+1)+(6x^2-3y^2-y-2z^2) = 3(x+z-2)(x-z).$$

Hence, $z = 2 - x$ or $z = x$. The case $z = 2 - x$ is impossible, because it would imply

$$x^2 + 2x + 1 - y = 6x - y + z^2 - 3 = 0 = x^2 - y^2 - 2z + 1 = x^2 + 2x - 3 - y^2;$$

hence $y - 1 = y^2 + 3$, that is,

$$(2y - 1)^2 = -15,$$

a contradiction. Thus $z = x$.

We obtain

$$0 = x^2 - y^2 - 2z + 1 = (x + y - 1)(x - y - 1),$$

so that $y = 1 - x$ or $y = x - 1$. If $y = 1 - x$, then

$$0 = 6x - y + z^2 - 3 = x^2 + 7x - 4,$$

which yields the first and the second solution given above. If $y = x - 1$ then

$$0 = 6x - y + z^2 - 3 = x^2 + 5x - 2,$$

and we get the third and the fourth solution.

OC352. Let O_1 and O_2 intersect at P and Q. Their common external tangent touches O_1 and O_2 at A and B respectively. A circle Γ passing through A and B intersects O_1, O_2 at D, C. Prove that

$$\frac{CP}{CQ} = \frac{DP}{DQ}.$$

Originally Problem 2 of Day 1 of 2016 China Western Mathematical Olympiad.

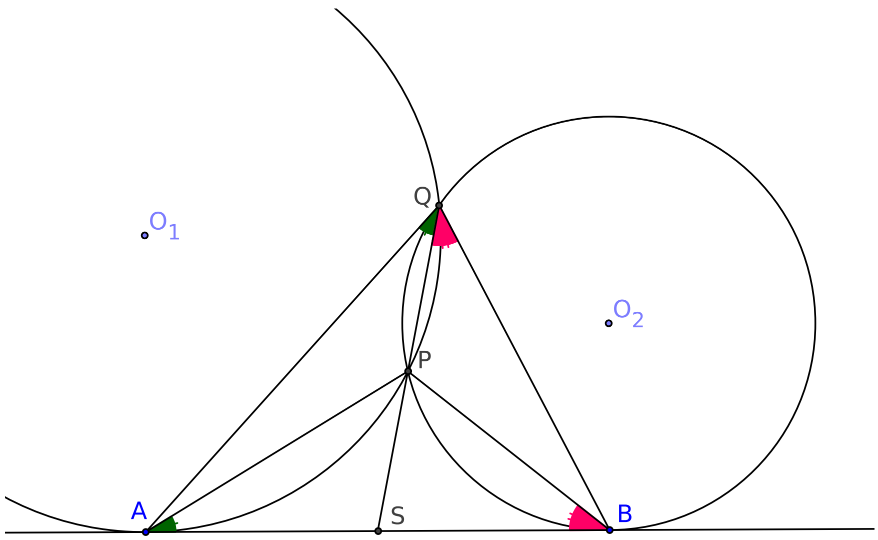
We received only one submission and we present the solution by Oliver Geupel.

The lines AB and PQ intersect at a point S which has equal powers with respect to the circles (O_1) and (O_2). It follows that $AS = BS$. We have

$$\angle BAP = \angle AQP = \angle AQS \quad \text{and} \quad \angle ABP = \angle BQP = \angle BQS$$

by the properties of inscribed angles. By the law of sines, we obtain

$$\frac{AP}{BP} = \frac{\sin\angle ABP}{\sin\angle BAP} = \frac{AS \cdot \dfrac{\sin\angle ASQ}{\sin\angle AQS}}{BS \cdot \dfrac{\sin\angle BSQ}{\sin\angle BQS}} = \frac{AQ}{BQ}. \tag{1}$$



Apply the inversion with centre at A and an arbitrary positive radius r; let X' denote the image of X. The circle (O_2) is transformed into a circle passing through C', P', B', Q'. By the properties of the inversion and by (1), we have

$$B'P' = \frac{r^2}{AB \cdot AP} \cdot BP = \frac{r^2}{AB \cdot AQ} \cdot BQ = B'Q'.$$

Hence, $C'B'$ bisects the angle $P'C'Q'$. The circle (O_1) with point A excluded is transformed into a line through P', D', Q'. The circle Γ without A is transformed into a line through C', D', B'. Thus, $C'D'$ is the internal bisector of angle C' in triangle $P'C'Q'$. By the properties of the inversion and by the angle bisector theorem, we obtain

$$\frac{CP}{CQ} = \frac{\dfrac{AC \cdot AP \cdot C'P'}{r^2}}{\dfrac{AC \cdot AQ \cdot C'Q'}{r^2}} = \frac{AP}{AQ} \cdot \frac{C'P'}{C'Q'} = \frac{AP}{AQ} \cdot \frac{D'P'}{D'Q'} = \frac{\dfrac{AD \cdot AP \cdot D'P'}{r^2}}{\dfrac{AD \cdot AQ \cdot D'Q'}{r^2}} = \frac{DP}{DQ}.$$

This completes the proof.

OC353. Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

Originally Problem 2 of Day 1 of 2016 USAMO. We received 2 solutions and present both of them.

Solution 1, by Ivko Dimitrić.

We proceed by induction. Let $N(k)$ denote the expression in the problem. Then $N(1) = 1$ and $N(2) = 4! \cdot \frac{0!}{2!} \cdot \frac{1!}{3!} = 2$ are integers. Assume $N(k)$ is an integer. Then,

$$\begin{aligned}
N(k+1) &= [(k+1)^2]! \cdot \prod_{j=0}^{k} \frac{j!}{(j+k+1)!} \\
&= [(k+1)^2]! \cdot \frac{k!}{(2k+1)!} \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k+1)(j+k)!} \\
&= [(k+1)^2]! \cdot \frac{k!}{(2k+1)!} \cdot \prod_{j=0}^{k-1} \frac{1}{j+k+1} \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \\
&= [(k+1)^2]! \cdot \frac{k!}{(2k+1)!} \cdot \frac{1}{(k+1)(k+2)\cdots(k+k)} \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \\
&= [(k+1)^2]! \cdot \frac{(k!)^2}{(2k+1)!\,(2k)!} N(k) \\
&= \frac{(2k+1+k^2)!\,(k!)^2}{(2k+1)!\,(2k)!} N(k) \\
&= \frac{(2k+2)(2k+3)\cdots(2k+k^2+1)}{(k+1)(k+2)\cdots(k+k)} (k!)\, N(k).
\end{aligned}$$

Since $k^2+1 > 2k$ for $k > 1$, then $2k+k^2+1 > 4k$ and the factors of the numerator of the last fraction include all doubled factors of the denominator,

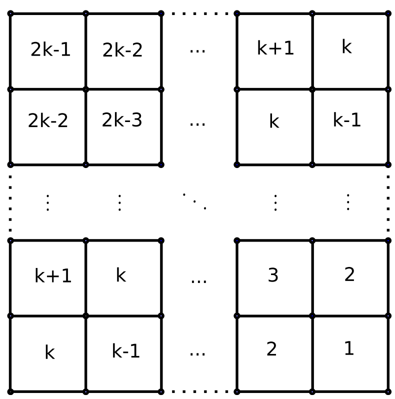
$$(2k+2) = 2(k+1),\ (2k+4) = 2(k+2),\ \cdots,\ 4k = 2(k+k),$$

so that the fraction is reduced to an integer quotient which is then multiplied by an integer $(k!)N(k)$ to produce integer value of $N(k+1)$, finishing the induction step and the proof.

Solution 2, by Oliver Geupel.

Let $n_1, n_2, \ldots, n_m$ be integers such that $n_1 \ge n_2 \ge \cdots \ge n_m > 0$, and let $N = n_1 + n_2 + \cdots + n_m$. A *Young tableau* of shape $(n_1, n_2, \ldots, n_m)$ is a collection of squares, arranged in m left-justified rows, with n_1 squares in the first row, n_2 squares in the second row, $\ldots$, n_m squares in the m-th row, in which the integers from 1 to N have to be inserted (one in each square) in such a way that all rows and columns are increasing. The *hook* corresponding to a square in a Young tableau is the union of the square and all squares to the right in the same row and the squares below in the same column. The *hook-length* is the number of squares in the hook.

Consider a Young tableau consisting of k rows of length k. The following figure shows in each square the corresponding hook-length:



The *hook length formula* expresses the number of Young tableaux of a given shape and a total of N squares as $N!$ divided by the product of all hook-lengths (see J.H. van Lint, R.M. Wilson, *A Course in Combinatorics*, 2nd ed., Cambridge University Press, 2001, Theorem 15.11, page 165).

By the hook length formula, the number of Young tableaux with k rows of length k is

$$\frac{(k^2)!}{(1\cdot 2\cdots k)\cdot(2\cdot 3\cdots(k+1))\cdots(k(k+1)\cdots(2k-1))} = (k^2)!\cdot\prod_{j=0}^{k-1}\frac{j!}{(j+k)!},$$

which is therefore an integer.

OC354. Solve the equation $1 + x^z + y^z = LCM(x^z, y^z)$ in the set of natural numbers.

Originally Problem 1 of 2016 Macedonia National Olympiad.

We received 4 solutions and will present 2 of them.

Solution 1, by Ivko Dimitrić.

Let $d = GCD(x, y)$ be the greatest common divisor of x and y. Then $x = dx_1,\ y = dy_1$, where $GCD(x_1, y_1) = 1$. Substituting these into the equation gives

$$1 + d^z x_1^z + d^z y_1^z = LCM(d^z x_1^z, d^z y_1^z) = d^z x_1^z y_1^z.$$

If $d > 1$ then all terms in this equation are divisible by d except 1, which is not possible. Thus, $GCD(x, y) = 1$ and $LCM(x^z, y^z) = x^z y^z$, so that the equation becomes

$$1 + x^z + y^z = x^z\, y^z,$$

which can be rewritten as $(x^z - 1)(y^z - 1) = 2$. In this product one of the factors is 1 and the other one is 2. If $x^z - 1 = 1$ and $y^z - 1 = 2$, we get $x = 2$, $y = 3$, $z = 1$, and if $x^z - 1 = 2$ and $y^z - 1 = 1$, we have $x = 3$, $y = 2$, $z = 1$, which are the only sets of solutions.

Solution 2, by the Missouri State University Problem Solving Group.

Suppose we have natural numbers x, y, z satisfying the equation. For simplicity, we consider the equation $1 + a + b = LCM(a, b)$, where $a = x^z$ and $b = y^z$. Since $1 + a + b$ is a multiple of a, we have a divides $1 + b$ and in particular $a \leq b + 1$. Similarly $b \leq a + 1$. Hence $a - 1 \leq b \leq a + 1$, so either (1) $b = a - 1$, (2) $b = a$, or (3) $b = a + 1$.

Note that in cases (1) and (3), a and b are relatively prime, so their least common multiple is their product.

The equation in case (1) reads $1 + a + a - 1 = a(a - 1)$, which gives $a = 3$, and then $b = 2$. The equation in case (2), $1 + a + a = a$ implies $a = -1$, which is not a natural number.

The equation in case (3) gives $1 + a + a + 1 = a(a + 1)$ whose only natural number solution is $a = 2$. Then $b = 3$ in this case.

Hence the only solutions in the natural numbers are $(x, y, z) = (3, 2, 1)$ and $(x, y, z) = (2, 3, 1)$.

Editor's Comments. Note that if we include zero in the set of natural numbers (as some authors in general do), it's easy to see that there are no solutions if $x = 0$ or $y = 0$ or $z = 0$ and the solutions are the ones already found.

OC355. Let $\mathbb{N}$ denote the set of natural numbers. Define a function $T : \mathbb{N} \to \mathbb{N}$ by $T(2k) = k$ and $T(2k + 1) = 2k + 2$. We write $T^2(n) = T(T(n))$ and in general $T^k(n) = T^{k-1}(T(n))$ for any $k > 1$.

a) Show that for each $n \in \mathbb{N}$, there exists k such that $T^k(n) = 1$.

b) For $k \in \mathbb{N}$, let c_k denote the number of elements in the set $\{n : T^k(n) = 1\}$. Prove that $c_{k+2} = c_{k+1} + c_k$, for $k \geq 1$.

Originally Problem 3 of 2016 India National Olympiad.

We only received 1 *submission and we present the solution by Oliver Geupel.*

For $n \in \mathbb{N}$, let $P(n)$ denote the assertion that there exists a natural number k with the property $T^k(n) = 1$. We prove $P(n)$ for all natural numbers n by mathematical induction. From $T^2(1) = 1$ we have $P(1)$. We prove $P(n)$ for $n > 1$ under the hypothesis that $P(m)$ holds for all natural numbers m less than the number n. In fact, if n is even, say $n = 2q$, then $P(n)$ follows from $T(n) = q$ and the hypothesis $P(q)$. If n is odd, say $n = 2q - 1$, then $P(n)$ follows from $T^2(n) = q$ and the hypothesis $P(q)$, because $q < n$. This completes the induction and the solution to part a).

Going on to part b), let $\#S$ denote the number of elements in the finite set S. By the recursion, we have for all natural numbers k and q:

$$T^{k+2}(2q) = 1 \Leftrightarrow T^{k+1}(q) = 1 \qquad \text{and} \qquad T^{k+2}(2q+1) = 1 \Leftrightarrow T^{k}(q) = 1.$$

Hence,

$$\begin{aligned} c_{k+2} &= \#\{n : T^{k+2}(n) = 1\} \\ &= \#\{q \in \mathbb{N} : T^{k+2}(2q) = 1\} + \#\{q \in \mathbb{N} : T^{k+2}(2q-1) = 1\} \\ &= \#\{q \in \mathbb{N} : T^{k+1}(q) = 1\} + \#\{q \in \mathbb{N} : T^{k}(q) = 1\} \qquad \text{(by recursion)} \\ &= c_{k+1} + c_k \end{aligned}$$

for all natural numbers k. This is the desired result b).