

# OLYMPIAD CORNER SOLUTIONS

*Statements of the problems in this section originally appear in 2018: 44(1), p. 13–15.*

**OC351.** Solve the system of equations

$$\begin{aligned}6x - y + z^2 &= 3, \\x^2 - y^2 - 2z &= -1, \\6x^2 - 3y^2 - y - 2z^2 &= 0.\end{aligned}$$

where  $x, y, z \in \mathbb{R}$ .

*Originally Problem 1 of Day 1 of 2016 Vietnam National Olympiad.*

*We received 7 submissions of which 6 were correct and complete. We present 2 solutions.*

*Solution 1, by Dan Daniel.*

First equation gives  $y = z^2 + 6x - 3$ ; second equation gives  $y^2 = x^2 - 2z + 1$ . Substituting these values into the third equation, we get

$$6x^2 - 3(x^2 - 2z + 1) - (z^2 + 6x - 3) - 2z^2 = 0 \implies 3x^2 - 3z^2 - 6x + 6z = 0,$$

i.e.

$$(x - z)(3(x + z) - 6) = 0 \implies (x - z)(x + z - 2) = 0.$$

We have two cases.

(i)  $x = 2 - z$ . Then, the first equation of the system becomes

$$6(2 - z) - y + z^3 = 3 \implies y = (z - 3)^2$$

and the second equation of the system becomes

$$(2 - z)^2 - (z - 3)^4 - 2z + 1 = 0 \implies (z - 3)^4 - (z - 3)^2 + 4 = 0.$$

Setting  $t = z - 3$ , we have no solutions in this case.

(ii)  $z = x$ . Then,  $y^2 = x^2 - 2x + 1$ , i.e.  $y = \pm(x - 1)$ . If  $y = x - 1$ , then from the first equation of the system, we get

$$x^2 + 5x - 2 = 0 \implies (x, y, z) = \left( \frac{-5 \pm \sqrt{33}}{2}, \frac{-7 \pm \sqrt{33}}{2}, \frac{-5 \pm \sqrt{33}}{2} \right).$$

If  $y = 1 - x$ , then from the first equation of the system, we get

$$x^2 + 7x - 4 = 0 \implies (x, y, z) = \left( \frac{-7 \pm \sqrt{65}}{2}, \frac{9 \mp \sqrt{65}}{2}, \frac{-7 \pm \sqrt{65}}{2} \right).$$

*Solution 2, by Oliver Geupel.*

It is tedious but straightforward to check that the following four triplets  $(x, y, z)$  are solutions to the problem:

$$\left(\frac{-7 + \sqrt{65}}{2}, \frac{9 - \sqrt{65}}{2}, \frac{-7 + \sqrt{65}}{2}\right), \left(\frac{-7 - \sqrt{65}}{2}, \frac{9 + \sqrt{65}}{2}, \frac{-7 - \sqrt{65}}{2}\right),$$

$$\left(\frac{-5 + \sqrt{33}}{2}, \frac{-7 + \sqrt{33}}{2}, \frac{-5 + \sqrt{33}}{2}\right), \left(\frac{-5 - \sqrt{33}}{2}, \frac{-7 - \sqrt{33}}{2}, \frac{-5 - \sqrt{33}}{2}\right).$$

We show that there is no further solution.

Suppose  $(x, y, z)$  is any solution. Then,

$$0 = (3 - 6x + y - z^2) - 3(x^2 - y^2 - 2z + 1) + (6x^2 - 3y^2 - y - 2z^2) = 3(x + z - 2)(x - z).$$

Hence,  $z = 2 - x$  or  $z = x$ . The case  $z = 2 - x$  is impossible, because it would imply

$$x^2 + 2x + 1 - y = 6x - y + z^2 - 3 = 0 = x^2 - y^2 - 2z + 1 = x^2 + 2x - 3 - y^2;$$

hence  $y - 1 = y^2 + 3$ , that is,

$$(2y - 1)^2 = -15,$$

a contradiction. Thus  $z = x$ .

We obtain

$$0 = x^2 - y^2 - 2z + 1 = (x + y - 1)(x - y - 1),$$

so that  $y = 1 - x$  or  $y = x - 1$ . If  $y = 1 - x$ , then

$$0 = 6x - y + z^2 - 3 = x^2 + 7x - 4,$$

which yields the first and the second solution given above. If  $y = x - 1$  then

$$0 = 6x - y + z^2 - 3 = x^2 + 5x - 2,$$

and we get the third and the fourth solution.

**OC352.** Let  $O_1$  and  $O_2$  intersect at  $P$  and  $Q$ . Their common external tangent touches  $O_1$  and  $O_2$  at  $A$  and  $B$  respectively. A circle  $\Gamma$  passing through  $A$  and  $B$  intersects  $O_1, O_2$  at  $D, C$ . Prove that

$$\frac{CP}{CQ} = \frac{DP}{DQ}.$$

*Originally Problem 2 of Day 1 of 2016 China Western Mathematical Olympiad.*

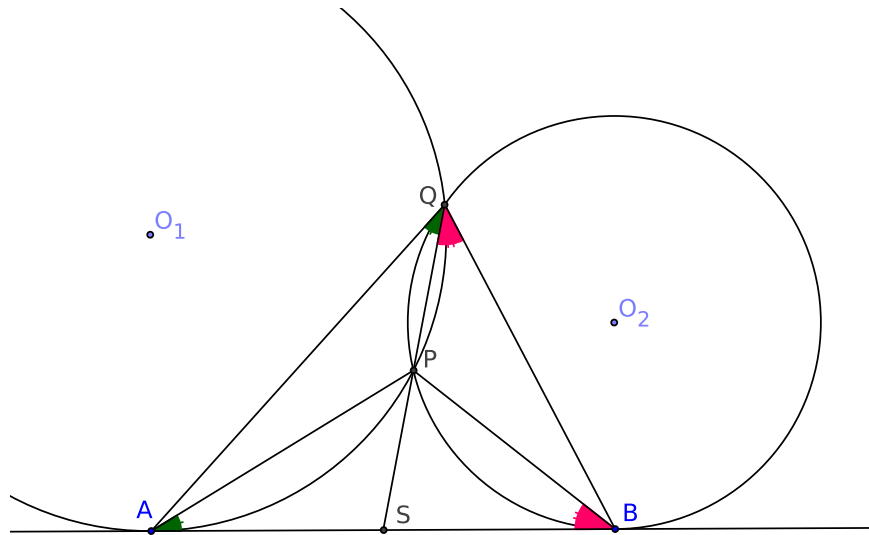
*We received only one submission and we present the solution by Oliver Geupel.*

The lines  $AB$  and  $PQ$  intersect at a point  $S$  which has equal powers with respect to the circles  $(O_1)$  and  $(O_2)$ . It follows that  $AS = BS$ . We have

$$\angle BAP = \angle AQP = \angle AQS \quad \text{and} \quad \angle ABP = \angle BQP = \angle BQS$$

by the properties of inscribed angles. By the law of sines, we obtain

$$\frac{AP}{BP} = \frac{\sin \angle ABP}{\sin \angle BAP} = \frac{AS \cdot \frac{\sin \angle ASQ}{\sin \angle AQS}}{BS \cdot \frac{\sin \angle BSQ}{\sin \angle BQS}} = \frac{AQ}{BQ}. \quad (1)$$



Apply the inversion with centre at  $A$  and an arbitrary positive radius  $r$ ; let  $X'$  denote the image of  $X$ . The circle  $(O_2)$  is transformed into a circle passing through  $C', P', B', Q'$ . By the properties of the inversion and by (1), we have

$$B'P' = \frac{r^2}{AB \cdot AP} \cdot BP = \frac{r^2}{AB \cdot AQ} \cdot BQ = B'Q'.$$

Hence,  $C'B'$  bisects the angle  $P'C'Q'$ . The circle  $(O_1)$  with point  $A$  excluded is transformed into a line through  $P', D', Q'$ . The circle  $\Gamma$  without  $A$  is transformed into a line through  $C', D', B'$ . Thus,  $C'D'$  is the internal bisector of angle  $C'$  in triangle  $P'C'Q'$ . By the properties of the inversion and by the angle bisector theorem, we obtain

$$\frac{CP}{CQ} = \frac{\frac{AC \cdot AP \cdot C'P'}{r^2}}{\frac{AC \cdot AQ \cdot C'Q'}{r^2}} = \frac{AP}{AQ} \cdot \frac{C'P'}{C'Q'} = \frac{AP}{AQ} \cdot \frac{D'P'}{D'Q'} = \frac{\frac{AD \cdot AP \cdot D'P'}{r^2}}{\frac{AD \cdot AQ \cdot D'Q'}{r^2}} = \frac{DP}{DQ}.$$

This completes the proof.

**OC353.** Prove that for any positive integer  $k$ ,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

*Originally Problem 2 of Day 1 of 2016 USAMO. We received 2 solutions and present both of them.*

*Solution 1, by Ivko Dimitrić.*

We proceed by induction. Let  $N(k)$  denote the expression in the problem. Then  $N(1) = 1$  and  $N(2) = 4! \cdot \frac{0!}{2!} \cdot \frac{1!}{3!} = 2$  are integers. Assume  $N(k)$  is an integer. Then,

$$\begin{aligned} N(k+1) &= [(k+1)^2]! \cdot \prod_{j=0}^k \frac{j!}{(j+k+1)!} \\ &= [(k+1)^2]! \cdot \frac{k!}{(2k+1)!} \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k+1)(j+k)!} \\ &= [(k+1)^2]! \cdot \frac{k!}{(2k+1)!} \cdot \prod_{j=0}^{k-1} \frac{1}{j+k+1} \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \\ &= [(k+1)^2]! \cdot \frac{k!}{(2k+1)!} \cdot \frac{1}{(k+1)(k+2)\cdots(k+k)} \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \\ &= [(k+1)^2]! \cdot \frac{(k!)^2}{(2k+1)!(2k)!} N(k) \\ &= \frac{(2k+1+k^2)!(k!)^2}{(2k+1)!(2k)!} N(k) \\ &= \frac{(2k+2)(2k+3)\cdots(2k+k^2+1)}{(k+1)(k+2)\cdots(k+k)} (k!) N(k). \end{aligned}$$

Since  $k^2+1 > 2k$  for  $k > 1$ , then  $2k+k^2+1 > 4k$  and the factors of the numerator of the last fraction include all doubled factors of the denominator,

$$(2k+2) = 2(k+1), (2k+4) = 2(k+2), \dots, 4k = 2(k+k),$$

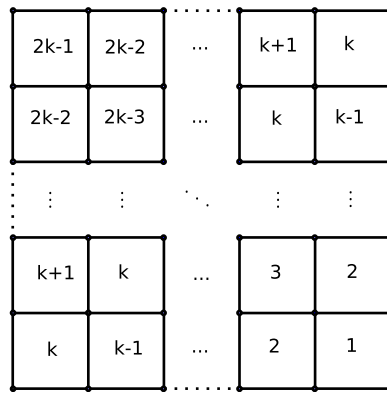
so that the fraction is reduced to an integer quotient which is then multiplied by an integer  $(k!)N(k)$  to produce integer value of  $N(k+1)$ , finishing the induction step and the proof.

*Solution 2, by Oliver Geupel.*

Let  $n_1, n_2, \dots, n_m$  be integers such that  $n_1 \geq n_2 \geq \dots \geq n_m > 0$ , and let  $N = n_1 + n_2 + \dots + n_m$ . A *Young tableau* of shape  $(n_1, n_2, \dots, n_m)$  is a collection of

squares, arranged in  $m$  left-justified rows, with  $n_1$  squares in the first row,  $n_2$  squares in the second row,  $\dots$ ,  $n_m$  squares in the  $m$ -th row, in which the integers from 1 to  $N$  have to be inserted (one in each square) in such a way that all rows and columns are increasing. The *hook* corresponding to a square in a Young tableau is the union of the square and all squares to the right in the same row and the squares below in the same column. The *hook-length* is the number of squares in the hook.

Consider a Young tableau consisting of  $k$  rows of length  $k$ . The following figure shows in each square the corresponding hook-length:



The *hook length formula* expresses the number of Young tableaux of a given shape and a total of  $N$  squares as  $N!$  divided by the product of all hook-lengths (see J.H. van Lint, R.M. Wilson, *A Course in Combinatorics*, 2nd ed., Cambridge University Press, 2001, Theorem 15.11, page 165).

By the hook length formula, the number of Young tableaux with  $k$  rows of length  $k$  is

$$\frac{(k^2)!}{(1 \cdot 2 \cdots k) \cdot (2 \cdot 3 \cdots (k+1)) \cdots (k(k+1) \cdots (2k-1))} = (k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!},$$

which is therefore an integer.

**OC354.** Solve the equation  $1 + x^z + y^z = LCM(x^z, y^z)$  in the set of natural numbers.

*Originally Problem 1 of 2016 Macedonia National Olympiad.*

*We received 4 solutions and will present 2 of them.*

*Solution 1, by Ivko Dimitrić.*

Let  $d = GCD(x, y)$  be the greatest common divisor of  $x$  and  $y$ . Then  $x = dx_1$ ,  $y = dy_1$ , where  $GCD(x_1, y_1) = 1$ . Substituting these into the equation gives

$$1 + d^z x_1^z + d^z y_1^z = LCM(d^z x_1^z, d^z y_1^z) = d^z x_1^z y_1^z.$$

If  $d > 1$  then all terms in this equation are divisible by  $d$  except 1, which is not possible. Thus,  $GCD(x, y) = 1$  and  $LCM(x^z, y^z) = x^z y^z$ , so that the equation becomes

$$1 + x^z + y^z = x^z y^z,$$

which can be rewritten as  $(x^z - 1)(y^z - 1) = 2$ . In this product one of the factors is 1 and the other one is 2. If  $x^z - 1 = 1$  and  $y^z - 1 = 2$ , we get  $x = 2, y = 3, z = 1$ , and if  $x^z - 1 = 2$  and  $y^z - 1 = 1$ , we have  $x = 3, y = 2, z = 1$ , which are the only sets of solutions.

*Solution 2, by the Missouri State University Problem Solving Group.*

Suppose we have natural numbers  $x, y, z$  satisfying the equation. For simplicity, we consider the equation  $1 + a + b = LCM(a, b)$ , where  $a = x^z$  and  $b = y^z$ . Since  $1 + a + b$  is a multiple of  $a$ , we have  $a$  divides  $1 + b$  and in particular  $a \leq b + 1$ . Similarly  $b \leq a + 1$ . Hence  $a - 1 \leq b \leq a + 1$ , so either (1)  $b = a - 1$ , (2)  $b = a$ , or (3)  $b = a + 1$ .

Note that in cases (1) and (3),  $a$  and  $b$  are relatively prime, so their least common multiple is their product.

The equation in case (1) reads  $1 + a + a - 1 = a(a - 1)$ , which gives  $a = 3$ , and then  $b = 2$ . The equation in case (2),  $1 + a + a = a$  implies  $a = -1$ , which is not a natural number.

The equation in case (3) gives  $1 + a + a + 1 = a(a + 1)$  whose only natural number solution is  $a = 2$ . Then  $b = 3$  in this case.

Hence the only solutions in the natural numbers are  $(x, y, z) = (3, 2, 1)$  and  $(x, y, z) = (2, 3, 1)$ .

*Editor's Comments.* Note that if we include zero in the set of natural numbers (as some authors in general do), it's easy to see that there are no solutions if  $x = 0$  or  $y = 0$  or  $z = 0$  and the solutions are the ones already found.

**OC355.** Let  $\mathbb{N}$  denote the set of natural numbers. Define a function  $T : \mathbb{N} \rightarrow \mathbb{N}$  by  $T(2k) = k$  and  $T(2k + 1) = 2k + 2$ . We write  $T^2(n) = T(T(n))$  and in general  $T^k(n) = T^{k-1}(T(n))$  for any  $k > 1$ .

- Show that for each  $n \in \mathbb{N}$ , there exists  $k$  such that  $T^k(n) = 1$ .
- For  $k \in \mathbb{N}$ , let  $c_k$  denote the number of elements in the set  $\{n : T^k(n) = 1\}$ . Prove that  $c_{k+2} = c_{k+1} + c_k$ , for  $k \geq 1$ .

*Originally Problem 3 of 2016 India National Olympiad.*

*We only received 1 submission and we present the solution by Oliver Geupel.*

For  $n \in \mathbb{N}$ , let  $P(n)$  denote the assertion that there exists a natural number  $k$  with the property  $T^k(n) = 1$ . We prove  $P(n)$  for all natural numbers  $n$  by mathematical induction. From  $T^2(1) = 1$  we have  $P(1)$ . We prove  $P(n)$  for  $n > 1$  under the hypothesis that  $P(m)$  holds for all natural numbers  $m$  less than the number  $n$ . In

fact, if  $n$  is even, say  $n = 2q$ , then  $P(n)$  follows from  $T(n) = q$  and the hypothesis  $P(q)$ . If  $n$  is odd, say  $n = 2q - 1$ , then  $P(n)$  follows from  $T^2(n) = q$  and the hypothesis  $P(q)$ , because  $q < n$ . This completes the induction and the solution to part a).

Going on to part b), let  $\#S$  denote the number of elements in the finite set  $S$ . By the recursion, we have for all natural numbers  $k$  and  $q$ :

$$T^{k+2}(2q) = 1 \Leftrightarrow T^{k+1}(q) = 1 \quad \text{and} \quad T^{k+2}(2q+1) = 1 \Leftrightarrow T^k(q) = 1.$$

Hence,

$$\begin{aligned} c_{k+2} &= \#\{n : T^{k+2}(n) = 1\} \\ &= \#\{q \in \mathbb{N} : T^{k+2}(2q) = 1\} + \#\{q \in \mathbb{N} : T^{k+2}(2q-1) = 1\} \\ &= \#\{q \in \mathbb{N} : T^{k+1}(q) = 1\} + \#\{q \in \mathbb{N} : T^k(q) = 1\} \quad (\text{by recursion}) \\ &= c_{k+1} + c_k \end{aligned}$$

for all natural numbers  $k$ . This is the desired result b).

