MathemAttic

No. 1

The problems featured in this section are intended for students at the secondary school level.

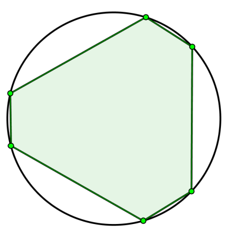
Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 15, 2019**.

MA1. How many two-digit numbers are there such that the difference of the number and the number with the digits reversed is a non-zero perfect square? Problem extension: What happens with three-digit numbers? four-digit numbers?

MA2. A sequence $t_1, t_2, \ldots$ beginning with any two positive numbers is defined such that for $n > 2$, $t_n = \frac{1+t_{n-1}}{t_{n-2}}$. Show that such a sequence must repeat itself with a period of 5.

MA3. A hexagon H is inscribed in a circle, and consists of three segments of length 1 and three segments of length 3. Find the area of H.



MA4. For what conditions on a and b is the line $x + y = a$ tangent to the circle $x^2 + y^2 = b$?

MA5. Point P lies in the first quadrant on the line $y = 2x$. Point Q is a point on the line $y = 3x$ such that PQ has length 5 and is perpendicular to the line $y = 2x$. Find the point P.

Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

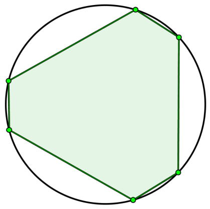
Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA1. Déterminer combien de nombres à deux chiffres sont tels que la différence entre le nombre et celui avec les chiffres renversés est un carré parfait. Généralisation du problème : qu'en est-il des nombres à trois chiffres ou à quatre chiffres ?

MA2. Une suite $t_1, t_2, \ldots$ débutant avec deux entiers positifs est telle que pour $n > 2$, on a $t_n = \frac{1+t_{n-1}}{t_{n-2}}$. Démontrer qu'une telle suite doit se répéter avec période 5.

MA3. Un hexagone H, dont trois côtés sont de longueur 1 et les trois autres de longueur 3, est inscrit dans un cercle. Déterminer la surface de cet hexagone.



MA4. Quelles conditions doit-on imposer à a et b de façon à ce que la ligne $x + y = a$ soit tangente au cercle $x^2 + y^2 = b$?

MA5. Un point P se situe dans le premier quadrant, sur la ligne $y = 2x$. Le point Q se situe sur la ligne $y = 3x$ et est tel que PQ est de longueur 5 et est perpendiculaire à la ligne $y = 2x$. Déterminer P.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(1), p. 4–5; and 44(2), p. 47–48.

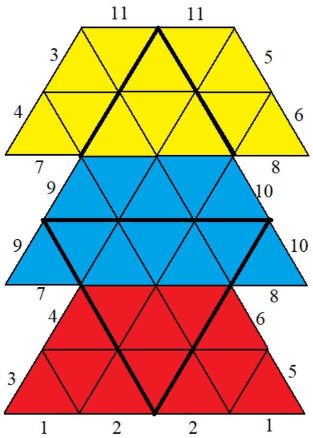
CC301. All natural numbers are coloured using 100 different colours. Prove that you can find several (no less than 2) different numbers, all of the same colour, that have a product with exactly 1000 different natural divisors.

Originally 2017 Savin Open Contest, Problem 7 (by E. Bakaev).

We received two solutions. We present the solution by Richard Hess.

Consider the set of numbers $p_1^9, p_2^9, \ldots, p_n^9$, where each p_i is a distinct prime and $n \geq 201$. If we colour each number in this set with any of 100 colours, then by the pigeonhole principle there will be at least three numbers with the same colour. The product of the three numbers has exactly 1000 natural divisors.

CC302. Nikolas used construction paper to make a regular tetrahedron (a pyramid consisting of equilateral triangles). Then he cut it in some ingenious way, unfolded it and this resulted in a Christmas tree-like shape consisting of three halves of a regular hexagon. How did Nikolas do this?



Originally 2017 Savin Open Contest, Problem 15 (by A. Domashenko).

We received one correct submission. We present the solution by the Missouri State University Problem Solving Group.

Folding the figure above along the heavily shaded lines and identifying edges labeled with the same number results in a regular tetrahedron.

CC303. Consider two convex polygons M and N with the following properties: polygon M has twice as many acute angles as polygon N has obtuse angles; polygon N has twice as many acute angles as polygon M has obtuse angles; each polygon has at least one acute angle; at least one of the polygons has a right angle.

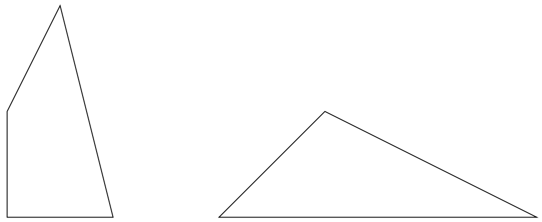
a) Give an example of such polygons.

b) How many right angles can each of these polygons have? Find the complete set of all the possibilities and prove that no others exist.

Originally 2017 Savin Open Contest, Problem 10 (by I. Akulich).

We received two solutions and present the slightly edited solution by Richard Hess.

If a convex polygon has at least one acute angle and the number of acute angles is even, then it has exactly two acute angles. Suppose it had four acute angles. Then each supplementary angle is greater than 90°, and thus their sum is greater than 360°, which is impossible for a convex polygon. Thus, M and N have each two acute angles and one obtuse angle. If either of M and N had two right angles, then the sum of the supplementary angles of the right angles and the acute angles would again be greater than 360°, which is impossible. We are left with two possibilities: quadrilaterals with one right angle, one obtuse angle and two acute angles, and triangles with one obtuse angle and two acute angles. M and N can be two quadrilaterals or one quadrilateral and one triangle.



CC304. Consider a natural number n greater than 1 and not divisible by 10. Can the last digit of n and second last digit of n^4 (that is, the digit in the tens position) be of the same parity?

Originally 2017 Savin Open Contest, Problem 1 (by S. Dvoryaninov).

We received 5 submissions, all of which were correct and complete. We present the solution by David Manes.

The answer is no. We begin by determining the last digit of n^2. Assume the digits of n are abc so that $n = 100a + 10b + c$, a is any non-negative integer, $c \neq 0$ since n is not divisible by 10. Then

$$\begin{aligned} n^2 = (100a + 10b + c)^2 &= 100(100a^2 + b^2 + 20ab + 2ac) + (20bc + c^2) \\ &\equiv 20bc + c^2 \pmod{100}. \end{aligned}$$

Consequently, the last two digits of n^2 are determined by the least residue modulo 100 of $20bc + c^2$. Note that a, the hundredths digit of n, has no bearing on the last two digits of n^2. More precisely, the last two digits of n^2 are the digits of t such that

$$20bc + c^2 \equiv t \pmod{100}.$$

The 21 possibilities are

$$01, 04, 09, 16, 21, 24, 25, 29, 36, 41, 44, 49, 56, 61, 64, 69, 76, 81, 84, 89, 96.$$

To find the last two digits of n^4, it suffices to find the least residue modulo 100 of

$$t^2 \equiv (20bc + c^2)^2 \equiv 40bc^3 + c^4 \pmod{100}.$$

Furthermore, note that if $b = 0, 1, 2, 3$, or 4, then

$$40(b+5)c^3 + c^4 \equiv 40bc^3 + 200c^3 + c^4 \equiv 40bc^3 + c^4 \pmod{100}.$$

The results are summarized in the following table, where $c \neq 0$ is the units digit of n and b is the tens digit of n for values of $b = 0, 1, 2, 3$ and 4.

b/c	1	2	3	4	5	6	7	8	9
0	01	16	81	56	25	96	01	96	61
1	41	36	61	16	25	36	21	76	21
2	81	56	41	76	25	76	41	56	81
3	21	76	21	36	25	16	61	36	41
4	61	96	01	96	25	56	81	16	01

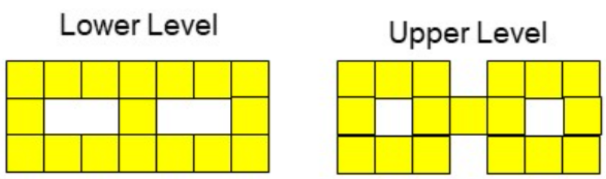
Therefore, the units digit of a positive integer n, not divisible by 10, is $c \neq 0$ which has the same parity as c^4, the units digit of n^4. The table then shows that in each case the tens digit of n^4 and c^4 have opposite parity. Accordingly, if n is not divisible by 10, then the units digit of n and the tens digit of n^4 have opposite parity.

CC305. Can you arrange n identical cubes in such a way that each cube has exactly three neighbours (cubes are considered to be neighbours if they have a common face)? Solve the problem for $n = 2016, 2017$ and 2018.

Originally 2017 Savin Open Contest, Problem 8 (by P. Kozhevnikov).

We received one submission. We present the solution by Richard Hess.

Putting 2016 or 2018 cubes together so each has exactly three neighbours can be done as shown below. Note that eight cubes in a two by two by two cube standing alone satisfies the requirement.

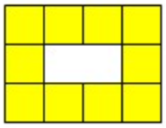


Case 1: 2016 *cubes.* Use 252 copies of the two by two by two cube.

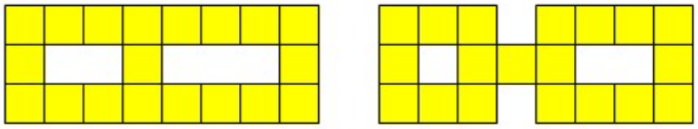
Case 2: 2018 *cubes.* Use 34 cubes in the lower and upper levels as shown along with 248 copies of the two by two by two cube.

Case 3: 2017 *cubes.* This is impossible. Consider an arrangement of any $2k+1$ cubes so each has exactly three neighbours. Then the number of shared faces is $F = 3(2k+1)/2$: three for each cube and divided by two because each shared face is counted twice. Expression for F is not an integer, producing a contradiction.

Additional thoughts. As shown in Case 2 above, we can get a solution for any $n \equiv 2 \pmod 8 \geq 34$ by adding copies of the two by two by two cube. We can get a solution for any $n \equiv 4 \pmod 8 \geq 20$ using two copies of the following frame, one above the other, and adding copies of the two by two by two cube:



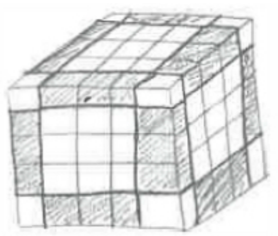
Finally, we can get a solution for any $n \equiv 6 \pmod 8 \geq 38$ using the two layers shown below and adding copies of the two by two by two cube:



CC306. Consider a $5 \times 5 \times 5$ cube with the outside surface painted blue. Buzz cuts the cube into 5^3 unit cubes, then picks a cube at random. Given that the cube Buzz picked has at least one painted blue face, what is the probability that the cube has exactly two blue faces?

Originally problem 6 of Ciphering round of 2016 Georgia Tech High School Mathematics Competition.

We received 11 submissions of which 9 were correct and complete. We present the solution by Kathleen E. Lewis, accompanied by an image submitted along with the solution by the Quest University Math Literacy Class.



Since there are 125 small cubes altogether, including 27 interior cubes, there are 98 cubes with at least one blue face. Each of the six faces contains 9 cubes with only one blue face, making 54 such cubes altogether. There are 8 corner cubes with three blue faces and 36 edge cubes with two blue faces (3 on each of the 12 edges). So the probability we want is $36/98 = 18/49$.

CC307. Find (with proof) all integer solutions (x, y) to

$$x^2 - xy + 2017y = 0.$$

Originally Problem 3 from the 2017 Science Atlantic Math Competition.

We received 9 complete solutions and 4 incomplete submissions. We present the solution by Richard Hess, slightly edited.

The equation

$$x^2 - xy + 2017y = 0$$

can be converted to

$$x^2 = y(x - 2017).$$

For $x = 2017$ the equation is not satisfied, so there is no solution in this case. Let $z = (x-2017)$ and the equation can be rewritten as $yz = (z+2017)^2$. From this it is clear that z must divide 2017^2. Because 2017 is prime, this happens only when $z = \pm 1, \pm 2017$, or $\pm 2017^2$. The solutions for x are thus

$$x = 2017 \cdot 2018, 2 \cdot 2017, 2018, 2016, 0 \text{ and } -2017 \cdot 2016$$

with the following corresponding solutions for y:

$$y = 2018^2, 4 \cdot 2017, 2018^2, -2016^2, 0 \text{ and } -2016^2.$$

CC308. Define the $n \times n$ *Pascal matrix* as follows: $a_{1j} = a_{i1} = 1$, while $a_{ij} = a_{i-1,j} + a_{i,j-i}$ for $i, j > 1$. So, for instance, the 3×3 Pascal matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

Show that every Pascal matrix is invertible.

Originally Problem 1 from the 2017 Science Atlantic Math Competition.

We received 9 correct solutions, out of which we present the one by Ángel Plaza, slightly edited.

Let P_n denote the $n \times n$ Pascal matrix. Then

$$P_n = [a_{ij}] = \left[\binom{i+j-1}{j-1}\right].$$

We will prove that $P_n = L_n \cdot U_n$, where L_n is a lower and U_n an upper triangular matrix with main diagonal equal to 1:

$$L_n = [l_{ij}] = \left[\binom{i-1}{j-1}\right] \quad \text{and} \quad U_n = [u_{ij}] = \left[\binom{j-1}{i-1}\right].$$

To prove the previous relations, it is enough to show that the i-th file of L_n multiplied by the j-th row of U_n is equal to the entry a_{ij} of P_n:

$$\sum_{k=1}^{n} l_{ik}u_{kj} = \sum_{k=0}^{n-1} \binom{i-1}{k}\binom{j-1}{k} \stackrel{*}{=} \sum_{k=0}^{j-1} \binom{i-1}{k}\binom{j-1}{j-1-k} = \binom{i+j-2}{-1} = a_{ij}.$$

Note that * in the above equation is Vandermonde's identity.

Thus $P_n = L_n \cdot U_n$ and therefore $\det(P_n) = \det(L_n) \cdot \det(U_n) = 1$.

CC309. Suppose $P(x)$ and $Q(x)$ are polynomials with real coefficients. Find necessary and sufficient conditions on N to guarantee that if the polynomial $P(Q(x))$ has degree N, there exists real x with $P(x) = Q(x)$.

Originally Problem 2 from the 2017 Science Atlantic Math Competition.

We received 4 solutions. We present the solution by Chun-Hao Huang.

We will show that the necessary and sufficient condition on N is that N is an odd positive integer which is not a perfect square. Note that there exists a real x with $P(x) = Q(x)$ if and only if $P(x) - Q(x) = 0$ has a real root. Denote by p and q the degrees of $P(x)$ and $Q(x)$ respectively; then, the degree N of $P(Q(x))$ is equal to pq.

Necessity. In the cases where N is even or an odd perfect square, one can find examples for $P(x)$ and $Q(x)$ for which $P(x) - Q(x)$ has no real root. Consider first the case when N is even. Choose

$$P(x) = x^N + x + 1 \quad \text{and} \quad Q(x) = x$$

(so $p = N$ and $q = 1$); then $P(x) - Q(x) = x^N + 1$, which has no real root. Next consider N an odd perfect square, and write $N = (k+1)^2$ with k even. Take

$$P(x) = x^{k+1} + x^k + 1 \quad \text{and} \quad Q(x) = x^{k+1}$$

(so $p = q = k + 1$); then $P(x) - Q(x) = x^k + 1$, which again has no real root.

Sufficiency. Suppose N is an odd integer which is not a perfect square. From $N = pq$ it follows that p and q are both odd, and $p \neq q$. The degree of $P(x)-Q(x)$ is then $\max\{p, q\}$, which is odd; thus $P(x) - Q(x)$ is a polynomial with real coefficients and odd degree, and as such must have at least one real root.

This concludes the proof that $P(x) = Q(x)$ has a real solution if and only if N is an odd positive integer which is not a perfect square.

CC310. Suppose

$$\tan x + \cot x + \sec x + \csc x = 6.$$

Find the value of

$$\sin x + \cos x.$$

Originally problem 8 of Ciphering round of 2016 Georgia Tech High School Mathematics Competition.

We received 9 submissions, of which eight are correct, and one cannot be opened. We present the solution provided by Maria Mateo and Ángel Plaza.

Notice that for the correctness of the given equation, $\sin x \neq 0$ and $\cos x \neq 0$. The given equation may be written as

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\sin x} + \frac{1}{\cos x} = 6,$$

or

$$1 + \sin x + \cos x = 6 \sin x \cos x. \tag{1}$$

Since

$$(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x,$$

then

$$\sin x \cos x = \frac{A^2 - 1}{2},$$

where $A = \sin x + \cos x$. Therefore (1) reads as

$$1 + A = 6\frac{A^2 - 1}{2},$$

which gives $A = -1$ or $A = 4/3$. Since $\sin x \neq 0$ and $\cos x \neq 0$, $A = -1$ is not a valid solution and so

$$A = \sin x + \cos x = 4/3.$$

PROBLEM SOLVING VIGNETTES

No.1

Shawn Godin

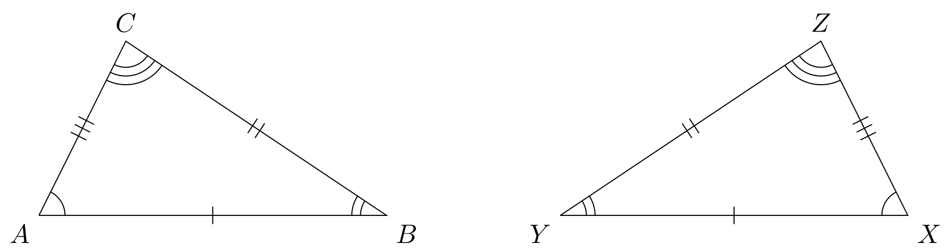
Congruent Triangles

Welcome *MathemAttic* readers! I am excited to see *Crux* reintroduce a section that is specifically aimed at a broader pre-university audience. This column will replace the column *Problem Solving 101* by expanding its scope. We will still, occasionally, focus on a particular problem and its solution. At other times, we will explore ideas from mathematics that will be of use to problem solvers.

Some of the ideas we will explore are things that have disappeared from the Canadian high school curriculum over the years. Other ideas will be things that students would run into later in high school or as an undergraduate, but that are accessible to younger readers without a lot of extra background. Over the years, in Canada anyway, Euclidean geometry has slowly all but disappeared from the curriculum. In this issue we will look at congruent triangles and their uses.

Two geometric figures in the plane are said to be *congruent* if they have the same size and shape; this means that they can be manipulated by use of translations, reflections, and rotations so that their corresponding parts coincide. For two triangles, we write $\triangle ABC \cong \triangle XYZ$ to indicate that triangles ABC and XYZ are congruent. If two triangles are congruent, their corresponding sides and angles are equal, so in our case we would have

$$\begin{aligned} AB &= XY & \angle A &= \angle X \\ BC &= YZ & \angle B &= \angle Y \\ CA &= ZX & \angle C &= \angle Z. \end{aligned}$$

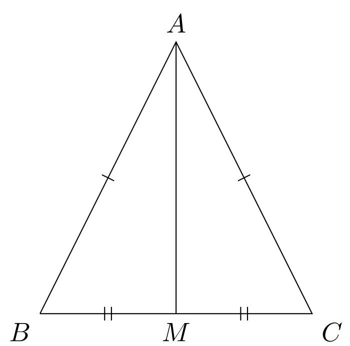


Notice that when we name the triangles in the congruence statement, the order of the vertices is important. The notation tells us how the vertices and sides match up.

We have several sufficient conditions for congruence:

- **side-side-side (SSS)** – when the three sides of one triangle equal the three sides of another (e.g. $AB = XY$, $BC = YZ$, and $CA = ZX$).
- **side-angle-side (SAS)** – when two sides and the *contained* angle of one triangle equal two sides and contained angle of another triangle (for example, $AB = XY$, $\angle B = \angle Y$, and $BC = YZ$). Note that angle-side-side is *not* a theorem — given ΔABC with $\angle A$ acute, there could be two triangles XYZ for which $\angle A = \angle X$, $AB = XY$, and $BC = YZ$.
- **angle-side-angle (ASA)** – when two angles and the *contained* side of one triangle equal two angles and the contained side of another triangle (e.g. $\angle A = \angle X$, $AB = XY$, and $\angle B = \angle Y$). Note if two angles and a non-contained side of one triangle are equal to those in another, then since the angles in a triangle add to $180°$, the third angles are equal as well, and we have ASA. Thus we can have ASA or AAS.
- **hypotenuse-side (HS)** – when the hypotenuse and a side of a right triangle are equal to the hypotenuse and side of another right triangle (e.g. $AB = XY$, $BC = YZ$, and $\angle C = 90° = \angle Z$).

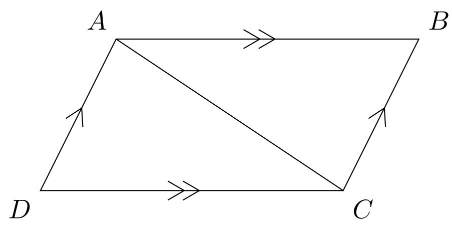
Let's see congruence in action. The word isosceles comes from Greek: *isos* meaning equal, and *skelos* meaning leg. Thus an isosceles triangle has (at least) two equal sides. Suppose $\triangle ABC$ is isosceles with $AB = AC$. Let M be the midpoint of BC and draw in the median AM as in the diagram below.



We now have two triangles $\triangle ABM$ and $\triangle ACM$. These triangles share the common side AM, have $AB = AC$ (since $\triangle ABC$ is isosceles) and $BM = CM$ since M is the midpoint of AB. Thus $\triangle ABM \cong \triangle ACM$ by SSS and the three angles must match up. Since $\angle B = \angle C$, we have shown that in any isosceles triangle, the angles opposite the equal sides are equal.

We can actually show more than that. From the congruence of the triangles we get that $\angle BAM = \angle CAM$, which means that AM bisects $\angle BAC$. Recall that AM was the median of $\triangle ABC$, so we have show that the median from the *apex*, the vertex between the two equal sides, also bisects the angle at the apex. Also, since $\angle BMA = \angle CMA$, by congruence, and $\angle BMA$ and $\angle CMA$ are supplementary, we can conclude that they are both right angles. Hence the altitude from the apex, the median from the apex and the angle bisector of the apex angle (the part not outside the triangle) all coincide.

Let's use congruent triangles to prove some other geometric properties. A *parallelogram* is a quadrilateral in which opposite sides are parallel. We will prove that the opposite sides are equal in length.



Let $ABCD$ be a parallelogram with $AB \parallel CD$ and $BC \parallel DA$. Draw in diagonal AC, creating two triangles. Since $AB \parallel CD$ then $\angle BAC = \angle DCA$ as they are alternate angles and similarly $\angle ACB = \angle CAD$. Finally, since the diagonal is a shared side of both triangles, $\triangle ABC \cong \triangle CDA$ by ASA, and hence $AB = CD$ and $BC = DA$.

You can also show that a quadrilateral where the opposite sides are equal in length is a parallelogram. This means that the conditions: "a quadrilateral has opposite sides equal in length" and "a quadrilateral has opposite sides that are parallel" are *equivalent conditions*. That means you cannot have one without the other as one implies the other.

The following are equivalent conditions on a quadrilateral $ABCD$:

1. $ABCD$ is a parallelogram (i.e. opposite sides are parallel),
2. opposite sides are equal in length,
3. opposite angles are equal to each other,
4. adjacent angles are supplementary,
5. one pair of opposite sides is equal in length and parallel,
6. the diagonals bisect each other,
7. the diagonals cut the quadrilateral into four triangles of equal area.

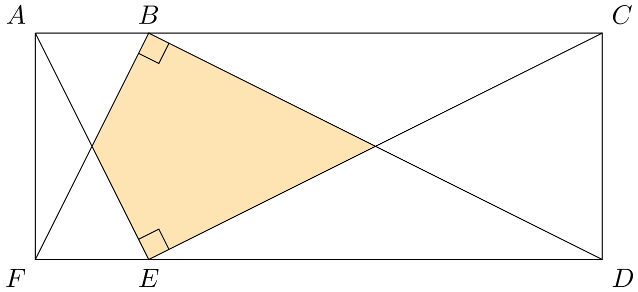
If you pick any two of these conditions you can show that they each imply the other. Also, you can number the conditions (1) to (7) in any order, and if you show that

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (1),$$

then every condition implies every other condition. For example, since $(1) \Rightarrow (2) \Rightarrow (3)$, we can conclude that $(1) \Rightarrow (3)$. Similarly, since $(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (1)$, we can conclude that $(3) \Rightarrow (1)$, hence $(1) \Leftrightarrow (3)$. I will leave these proofs as an exercise, some will use congruent triangles while others will not.

We will end with one more example. The following problem is inspired by problem 7a from the *2015 Euclid Contest*:

> In the diagram, $ACDF$ is a rectangle. Also, $\triangle FBD$ and $\triangle AEC$ are congruent triangles which are right-angled at B and E, respectively. Show that the shaded area is one quarter the area of the rectangle.



As $\triangle FBD \cong \triangle AEC$ we have $FB = AE$. So in the two right angled triangles $\triangle AFE$ and $\triangle FAB$, their hypotenuses are equal and AF is a shared side so they are congruent by HS, thus $AB = FE$. Since $ACDF$ is a rectangle, $AC \parallel DF$ and so $\angle BAE = \angle FEA$ as they are alternate angles. Therefore, $\triangle ABE \cong \triangle EFA$ by SAS as we have already shown that $AB = EF$ and clearly $AE = EA$. It follows that $\angle EBA = \angle AFE = 90°$. Thus $ABEF$ and, similarly, $BCDE$ are both rectangles. Hence, by the last property in the list of equivalent properties of a parallelogram, one quarter of each rectangle is shaded, and thus one quarter of the whole figure is shaded.

Euclidean geometry offers many opportunities to play and yet is lightly treated in current curricula. We will explore other ideas from Euclidean geometry in future columns. Here are some practice problems:

1. Line segments AB and CD bisect each other at E. Prove that $AC = BD$.
2. Given line segment AB, draw two circles of equal radii centred at A and B such that the two circles intersect at two points X and Y. Prove that AB and XY are perpendicular and bisect each other.
3. Point P is in the interior of $\triangle ABC$ such that it is equidistant from the three sides of the triangle. Prove that P lies on the three angle bisectors of the triangle.