CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(1), p. 4–5; and 44(2), p. 47–48.

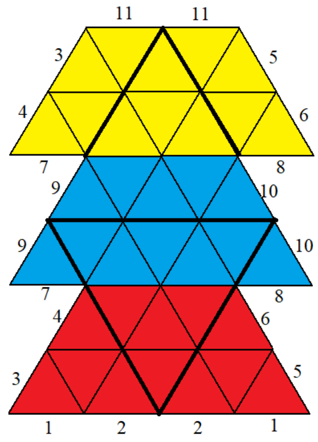
CC301. All natural numbers are coloured using 100 different colours. Prove that you can find several (no less than 2) different numbers, all of the same colour, that have a product with exactly 1000 different natural divisors.

Originally 2017 Savin Open Contest, Problem 7 (by E. Bakaev).

We received two solutions. We present the solution by Richard Hess.

Consider the set of numbers $p_1^9, p_2^9, \ldots, p_n^9$, where each p_i is a distinct prime and $n \geq 201$. If we colour each number in this set with any of 100 colours, then by the pigeonhole principle there will be at least three numbers with the same colour. The product of the three numbers has exactly 1000 natural divisors.

CC302. Nikolas used construction paper to make a regular tetrahedron (a pyramid consisting of equilateral triangles). Then he cut it in some ingenious way, unfolded it and this resulted in a Christmas tree-like shape consisting of three halves of a regular hexagon. How did Nikolas do this?



Originally 2017 Savin Open Contest, Problem 15 (by A. Domashenko).

We received one correct submission. We present the solution by the Missouri State University Problem Solving Group.

Folding the figure above along the heavily shaded lines and identifying edges labeled with the same number results in a regular tetrahedron.

CC303. Consider two convex polygons M and N with the following properties: polygon M has twice as many acute angles as polygon N has obtuse angles; polygon N has twice as many acute angles as polygon M has obtuse angles; each polygon has at least one acute angle; at least one of the polygons has a right angle.

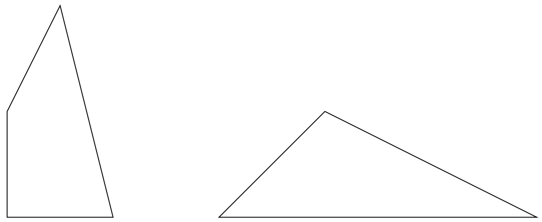
a) Give an example of such polygons.

b) How many right angles can each of these polygons have? Find the complete set of all the possibilities and prove that no others exist.

Originally 2017 Savin Open Contest, Problem 10 (by I. Akulich).

We received two solutions and present the slightly edited solution by Richard Hess.

If a convex polygon has at least one acute angle and the number of acute angles is even, then it has exactly two acute angles. Suppose it had four acute angles. Then each supplementary angle is greater than $90°$, and thus their sum is greater than $360°$, which is impossible for a convex polygon. Thus, M and N have each two acute angles and one obtuse angle. If either of M and N had two right angles, then the sum of the supplementary angles of the right angles and the acute angles would again be greater than $360°$, which is impossible. We are left with two possibilities: quadrilaterals with one right angle, one obtuse angle and two acute angles, and triangles with one obtuse angle and two acute angles. M and N can be two quadrilaterals or one quadrilateral and one triangle.



CC304. Consider a natural number n greater than 1 and not divisible by 10. Can the last digit of n and second last digit of n^4 (that is, the digit in the tens position) be of the same parity?

Originally 2017 Savin Open Contest, Problem 1 (by S. Dvoryaninov).

We received 5 submissions, all of which were correct and complete. We present the solution by David Manes.

The answer is no. We begin by determining the last digit of n^2. Assume the digits of n are abc so that $n = 100a + 10b + c$, a is any non-negative integer, $c \neq 0$ since n is not divisible by 10. Then

$$\begin{aligned} n^2 = (100a + 10b + c)^2 &= 100(100a^2 + b^2 + 20ab + 2ac) + (20bc + c^2) \\ &\equiv 20bc + c^2 \pmod{100}. \end{aligned}$$

Consequently, the last two digits of n^2 are determined by the least residue modulo 100 of $20bc + c^2$. Note that a, the hundredths digit of n, has no bearing on the last two digits of n^2. More precisely, the last two digits of n^2 are the digits of t such that

$$20bc + c^2 \equiv t \pmod{100}.$$

The 21 possibilities are

$$01, 04, 09, 16, 21, 24, 25, 29, 36, 41, 44, 49, 56, 61, 64, 69, 76, 81, 84, 89, 96.$$

To find the last two digits of n^4, it suffices to find the least residue modulo 100 of

$$t^2 \equiv (20bc + c^2)^2 \equiv 40bc^3 + c^4 \pmod{100}.$$

Furthermore, note that if $b = 0, 1, 2, 3,$ or 4, then

$$40(b+5)c^3 + c^4 \equiv 40bc^3 + 200c^3 + c^4 \equiv 40bc^3 + c^4 \pmod{100}.$$

The results are summarized in the following table, where $c \neq 0$ is the units digit of n and b is the tens digit of n for values of $b = 0, 1, 2, 3$ and 4.

b/c	1	2	3	4	5	6	7	8	9
0	01	16	81	56	25	96	01	96	61
1	41	36	61	16	25	36	21	76	21
2	81	56	41	76	25	76	41	56	81
3	21	76	21	36	25	16	61	36	41
4	61	96	01	96	25	56	81	16	01

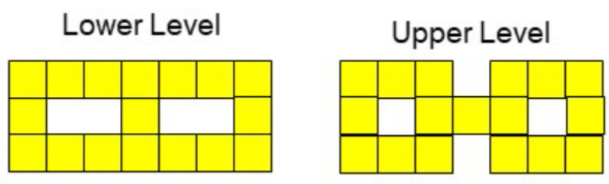
Therefore, the units digit of a positive integer n, not divisible by 10, is $c \neq 0$ which has the same parity as c^4, the units digit of n^4. The table then shows that in each case the tens digit of n^4 and c^4 have opposite parity. Accordingly, if n is not divisible by 10, then the units digit of n and the tens digit of n^4 have opposite parity.

CC305. Can you arrange n identical cubes in such a way that each cube has exactly three neighbours (cubes are considered to be neighbours if they have a common face)? Solve the problem for $n = 2016, 2017$ and 2018.

Originally 2017 Savin Open Contest, Problem 8 (by P. Kozhevnikov).

We received one submission. We present the solution by Richard Hess.

Putting 2016 or 2018 cubes together so each has exactly three neighbours can be done as shown below. Note that eight cubes in a two by two by two cube standing alone satisfies the requirement.

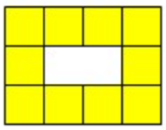


Case 1: 2016 *cubes.* Use 252 copies of the two by two by two cube.

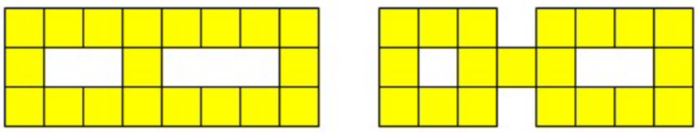
Case 2: 2018 *cubes.* Use 34 cubes in the lower and upper levels as shown along with 248 copies of the two by two by two cube.

Case 3: 2017 *cubes.* This is impossible. Consider an arrangement of any $2k+1$ cubes so each has exactly three neighbours. Then the number of shared faces is $F = 3(2k+1)/2$: three for each cube and divided by two because each shared face is counted twice. Expression for F is not an integer, producing a contradiction.

Additional thoughts. As shown in Case 2 above, we can get a solution for any $n \equiv 2 \pmod 8 \geq 34$ by adding copies of the two by two by two cube. We can get a solution for any $n \equiv 4 \pmod 8 \geq 20$ using two copies of the following frame, one above the other, and adding copies of the two by two by two cube:



Finally, we can get a solution for any $n \equiv 6 \pmod 8 \geq 38$ using the two layers shown below and adding copies of the two by two by two cube:

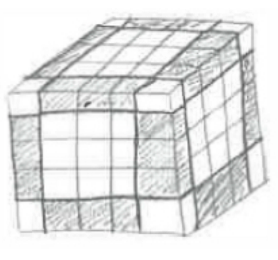


CC306. Consider a $5 \times 5 \times 5$ cube with the outside surface painted blue. Buzz cuts the cube into 5^3 unit cubes, then picks a cube at random. Given that the cube Buzz picked has at least one painted blue face, what is the probability that the cube has exactly two blue faces?

Originally problem 6 of Ciphering round of 2016 Georgia Tech High School Mathematics Competition.

We received 11 submissions of which 9 were correct and complete. We present the solution by Kathleen E. Lewis, accompanied by an image submitted along with the solution by the Quest University Math Literacy Class.

Since there are 125 small cubes altogether, including 27 interior cubes, there are 98 cubes with at least one blue face. Each of the six faces contains 9 cubes with only one blue face, making 54 such cubes altogether. There are 8 corner cubes with three blue faces and 36 edge cubes with two blue faces (3 on each of the 12 edges). So the probability we want is $36/98 = 18/49$.



CC307. Find (with proof) all integer solutions (x, y) to

$$x^2 - xy + 2017y = 0.$$

Originally Problem 3 from the 2017 Science Atlantic Math Competition.

We received 9 complete solutions and 4 incomplete submissions. We present the solution by Richard Hess, slightly edited.

The equation

$$x^2 - xy + 2017y = 0$$

can be converted to

$$x^2 = y(x - 2017).$$

For $x = 2017$ the equation is not satisfied, so there is no solution in this case. Let $z = (x - 2017)$ and the equation can be rewritten as $yz = (z + 2017)^2$. From this it is clear that z must divide 2017^2. Because 2017 is prime, this happens only when $z = \pm 1, \pm 2017$, or $\pm 2017^2$. The solutions for x are thus

$$x = 2017 \cdot 2018, 2 \cdot 2017, 2018, 2016, 0 \text{ and } -2017 \cdot 2016$$

with the following corresponding solutions for y:

$$y = 2018^2, 4 \cdot 2017, 2018^2, -2016^2, 0 \text{ and } -2016^2.$$

CC308. Define the $n \times n$ *Pascal matrix* as follows: $a_{1j} = a_{i1} = 1$, while $a_{ij} = a_{i-1,j} + a_{i,j-i}$ for $i, j > 1$. So, for instance, the 3×3 Pascal matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

Show that every Pascal matrix is invertible.

Originally Problem 1 from the 2017 Science Atlantic Math Competition.

We received 9 correct solutions, out of which we present the one by Ángel Plaza, slightly edited.

Let P_n denote the $n \times n$ Pascal matrix. Then

$$P_n = [a_{ij}] = \left[\binom{i+j-1}{j-1} \right].$$

We will prove that $P_n = L_n \cdot U_n$, where L_n is a lower and U_n an upper triangular matrix with main diagonal equal to 1:

$$L_n = [l_{ij}] = \left[\binom{i-1}{j-1} \right] \quad \text{and} \quad U_n = [u_{ij}] = \left[\binom{j-1}{i-1} \right].$$

To prove the previous relations, it is enough to show that the i-th file of L_n multiplied by the j-th row of U_n is equal to the entry a_{ij} of P_n:

$$\begin{aligned}\sum_{k=1}^{n} l_{ik}u_{kj} &= \sum_{k=0}^{n-1}\binom{i-1}{k}\binom{j-1}{k}\\ &\overset{*}{=} \sum_{k=0}^{j-1}\binom{i-1}{k}\binom{j-1}{j-1-k}\\ &= \binom{i+j-2}{-1} = a_{ij}.\end{aligned}$$

Note that * in the above equation is Vandermonde's identity.

Thus $P_n = L_n \cdot U_n$ and therefore $\det(P_n) = \det(L_n)\cdot\det(U_n) = 1$.

CC309. Suppose $P(x)$ and $Q(x)$ are polynomials with real coefficients. Find necessary and sufficient conditions on N to guarantee that if the polynomial $P(Q(x))$ has degree N, there exists real x with $P(x) = Q(x)$.

Originally Problem 2 from the 2017 Science Atlantic Math Competition.

We received 4 solutions. We present the solution by Chun-Hao Huang.

We will show that the necessary and sufficient condition on N is that N is an odd positive integer which is not a perfect square. Note that there exists a real x with $P(x) = Q(x)$ if and only if $P(x) - Q(x) = 0$ has a real root. Denote by p and q the degrees of $P(x)$ and $Q(x)$ respectively; then, the degree N of $P(Q(x))$ is equal to pq.

Necessity. In the cases where N is even or an odd perfect square, one can find examples for $P(x)$ and $Q(x)$ for which $P(x) - Q(x)$ has no real root. Consider first the case when N is even. Choose

$$P(x) = x^N + x + 1 \quad\text{and}\quad Q(x) = x$$

(so $p = N$ and $q = 1$); then $P(x) - Q(x) = x^N + 1$, which has no real root. Next consider N an odd perfect square, and write $N = (k+1)^2$ with k even. Take

$$P(x) = x^{k+1} + x^k + 1 \quad\text{and}\quad Q(x) = x^{k+1}$$

(so $p = q = k+1$); then $P(x) - Q(x) = x^k + 1$, which again has no real root.

Sufficiency. Suppose N is an odd integer which is not a perfect square. From $N = pq$ it follows that p and q are both odd, and $p \neq q$. The degree of $P(x)-Q(x)$ is then $\max\{p,q\}$, which is odd; thus $P(x) - Q(x)$ is a polynomial with real coefficients and odd degree, and as such must have at least one real root.

This concludes the proof that $P(x) = Q(x)$ has a real solution if and only if N is an odd positive integer which is not a perfect square.

CC310. Suppose

$$\tan x + \cot x + \sec x + \csc x = 6.$$

Find the value of

$$\sin x + \cos x.$$

Originally problem 8 of Ciphering round of 2016 Georgia Tech High School Mathematics Competition.

We received 9 submissions, of which eight are correct, and one cannot be opened. We present the solution provided by Maria Mateo and Ángel Plaza.

Notice that for the correctness of the given equation, $\sin x \neq 0$ and $\cos x \neq 0$. The given equation may be written as

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\sin x} + \frac{1}{\cos x} = 6,$$

or

$$1 + \sin x + \cos x = 6 \sin x \cos x. \tag{1}$$

Since

$$(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x,$$

then

$$\sin x \cos x = \frac{A^2 - 1}{2},$$

where $A = \sin x + \cos x$. Therefore (1) reads as

$$1 + A = 6 \frac{A^2 - 1}{2},$$

which gives $A = -1$ or $A = 4/3$. Since $\sin x \neq 0$ and $\cos x \neq 0$, $A = -1$ is not a valid solution and so

$$A = \sin x + \cos x = 4/3.$$