PROBLEM SOLVING VIGNETTES

No.1

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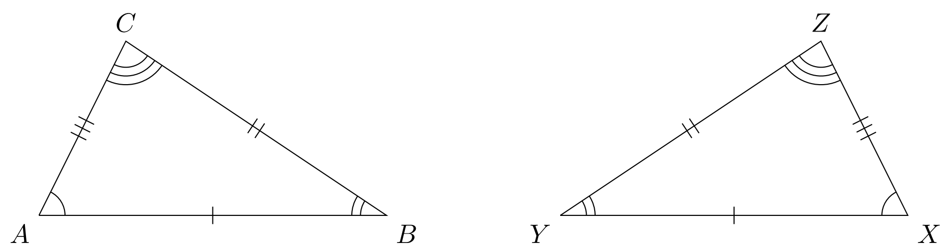
Congruent Triangles

Welcome *MathemAttic* readers! I am excited to see *Crux* reintroduce a section that is specifically aimed at a broader pre-university audience. This column will replace the column *Problem Solving 101* by expanding its scope. We will still, occasionally, focus on a particular problem and its solution. At other times, we will explore ideas from mathematics that will be of use to problem solvers.

Some of the ideas we will explore are things that have disappeared from the Canadian high school curriculum over the years. Other ideas will be things that students would run into later in high school or as an undergraduate, but that are accessible to younger readers without a lot of extra background. Over the years, in Canada anyway, Euclidean geometry has slowly all but disappeared from the curriculum. In this issue we will look at congruent triangles and their uses.

Two geometric figures in the plane are said to be *congruent* if they have the same size and shape; this means that they can be manipulated by use of translations, reflections, and rotations so that their corresponding parts coincide. For two triangles, we write $\triangle ABC \cong \triangle XYZ$ to indicate that triangles ABC and XYZ are congruent. If two triangles are congruent, their corresponding sides and angles are equal, so in our case we would have

$$\begin{aligned} AB &= XY & \qquad \angle A &= \angle X \\ BC &= YZ & \angle B &= \angle Y \\ CA &= ZX & \angle C &= \angle Z. \end{aligned}$$

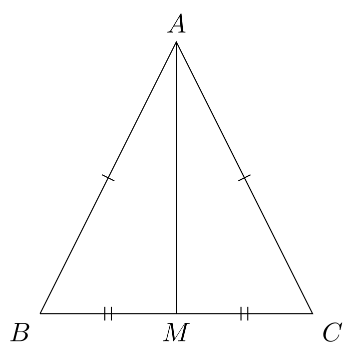


Notice that when we name the triangles in the congruence statement, the order of the vertices is important. The notation tells us how the vertices and sides match up.

We have several sufficient conditions for congruence:

- **side-side-side (SSS)** – when the three sides of one triangle equal the three sides of another (e.g. $AB = XY$, $BC = YZ$, and $CA = ZX$).
- **side-angle-side (SAS)** – when two sides and the *contained* angle of one triangle equal two sides and contained angle of another triangle (for example, $AB = XY$, $\angle B = \angle Y$, and $BC = YZ$). Note that angle-side-side is *not* a theorem — given ΔABC with $\angle A$ acute, there could be two triangles XYZ for which $\angle A = \angle X$, $AB = XY$, and $BC = YZ$.
- **angle-side-angle (ASA)** – when two angles and the *contained* side of one triangle equal two angles and the contained side of another triangle (e.g. $\angle A = \angle X$, $AB = XY$, and $\angle B = \angle Y$). Note if two angles and a non-contained side of one triangle are equal to those in another, then since the angles in a triangle add to $180°$, the third angles are equal as well, and we have ASA. Thus we can have ASA or AAS.
- **hypotenuse-side (HS)** – when the hypotenuse and a side of a right triangle are equal to the hypotenuse and side of another right triangle (e.g. $AB = XY$, $BC = YZ$, and $\angle C = 90° = \angle Z$).

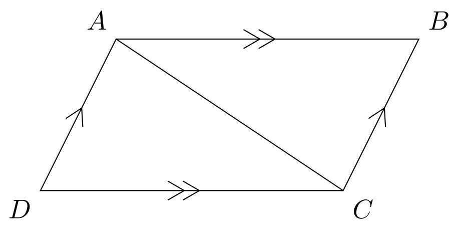
Let's see congruence in action. The word isosceles comes from Greek: *isos* meaning equal, and *skelos* meaning leg. Thus an isosceles triangle has (at least) two equal sides. Suppose $\triangle ABC$ is isosceles with $AB = AC$. Let M be the midpoint of BC and draw in the median AM as in the diagram below.



We now have two triangles $\triangle ABM$ and $\triangle ACM$. These triangles share the common side AM, have $AB = AC$ (since $\triangle ABC$ is isosceles) and $BM = CM$ since M is the midpoint of AB. Thus $\triangle ABM \cong \triangle ACM$ by SSS and the three angles must match up. Since $\angle B = \angle C$, we have shown that in any isosceles triangle, the angles opposite the equal sides are equal.

We can actually show more than that. From the congruence of the triangles we get that $\angle BAM = \angle CAM$, which means that AM bisects $\angle BAC$. Recall that AM was the median of $\triangle ABC$, so we have show that the median from the *apex*, the vertex between the two equal sides, also bisects the angle at the apex. Also, since $\angle BMA = \angle CMA$, by congruence, and $\angle BMA$ and $\angle CMA$ are supplementary, we can conclude that they are both right angles. Hence the altitude from the apex, the median from the apex and the angle bisector of the apex angle (the part not outside the triangle) all coincide.

Let's use congruent triangles to prove some other geometric properties. A *parallelogram* is a quadrilateral in which opposite sides are parallel. We will prove that the opposite sides are equal in length.



Let $ABCD$ be a parallelogram with $AB \parallel CD$ and $BC \parallel DA$. Draw in diagonal AC, creating two triangles. Since $AB \parallel CD$ then $\angle BAC = \angle DCA$ as they are alternate angles and similarly $\angle ACB = \angle CAD$. Finally, since the diagonal is a shared side of both triangles, $\triangle ABC \cong \triangle CDA$ by ASA, and hence $AB = CD$ and $BC = DA$.

You can also show that a quadrilateral where the opposite sides are equal in length is a parallelogram. This means that the conditions: "a quadrilateral has opposite sides equal in length" and "a quadrilateral has opposite sides that are parallel" are *equivalent conditions*. That means you cannot have one without the other as one implies the other.

The following are equivalent conditions on a quadrilateral $ABCD$:

1. $ABCD$ is a parallelogram (i.e. opposite sides are parallel),
2. opposite sides are equal in length,
3. opposite angles are equal to each other,
4. adjacent angles are supplementary,
5. one pair of opposite sides is equal in length and parallel,
6. the diagonals bisect each other,
7. the diagonals cut the quadrilateral into four triangles of equal area.

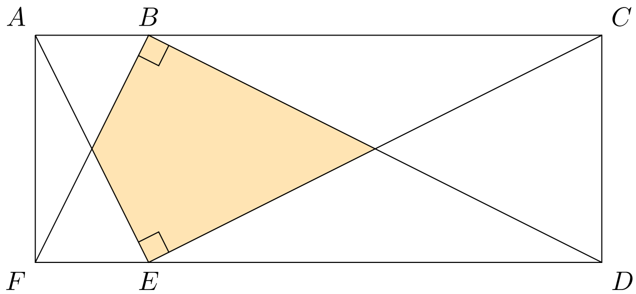
If you pick any two of these conditions you can show that they each imply the other. Also, you can number the conditions (1) to (7) in any order, and if you show that

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (1),$$

then every condition implies every other condition. For example, since $(1) \Rightarrow (2) \Rightarrow (3)$, we can conclude that $(1) \Rightarrow (3)$. Similarly, since $(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (1)$, we can conclude that $(3) \Rightarrow (1)$, hence $(1) \Leftrightarrow (3)$. I will leave these proofs as an exercise, some will use congruent triangles while others will not.

We will end with one more example. The following problem is inspired by problem 7a from the *2015 Euclid Contest*:

> In the diagram, $ACDF$ is a rectangle. Also, $\triangle FBD$ and $\triangle AEC$ are congruent triangles which are right-angled at B and E, respectively. Show that the shaded area is one quarter the area of the rectangle.



As $\triangle FBD \cong \triangle AEC$ we have $FB = AE$. So in the two right angled triangles $\triangle AFE$ and $\triangle FAB$, their hypotenuses are equal and AF is a shared side so they are congruent by HS, thus $AB = FE$. Since $ACDF$ is a rectangle, $AC \parallel DF$ and so $\angle BAE = \angle FEA$ as they are alternate angles. Therefore, $\triangle ABE \cong \triangle EFA$ by SAS as we have already shown that $AB = EF$ and clearly $AE = EA$. It follows that $\angle EBA = \angle AFE = 90°$. Thus $ABEF$ and, similarly, $BCDE$ are both rectangles. Hence, by the last property in the list of equivalent properties of a parallelogram, one quarter of each rectangle is shaded, and thus one quarter of the whole figure is shaded.

Euclidean geometry offers many opportunities to play and yet is lightly treated in current curricula. We will explore other ideas from Euclidean geometry in future columns. Here are some practice problems:

1. Line segments AB and CD bisect each other at E. Prove that $AC = BD$.
2. Given line segment AB, draw two circles of equal radii centred at A and B such that the two circles intersect at two points X and Y. Prove that AB and XY are perpendicular and bisect each other.
3. Point P is in the interior of $\triangle ABC$ such that it is equidistant from the three sides of the triangle. Prove that P lies on the three angle bisectors of the triangle.