FOCUS ON...

No. 34

Michel Bataille

Some Trigonometric Relations

Introduction

Trigonometry is quite a vast topic! In this number, we will just consider a selection of problems involving the values of the circular functions at $\frac{m\pi}{n}$ for various natural numbers m, n. Solving these problems calls for a perfect knowledge of the classical trigonometric formulas, of course, but other subjects such as complex numbers or polynomials can often prove useful. We will see various techniques at work in the examples that follow.

Juggling with trig formulas

Let us start gently with two exercises that I used to set to my students:

Evaluate a) $A = \sin^2 \dfrac{\pi}{9} + \sin^2 \dfrac{2\pi}{9} + \sin^2 \dfrac{4\pi}{9}$

b) $B = \sin \dfrac{\pi}{30} - \sin \dfrac{7\pi}{30} - \sin \dfrac{11\pi}{30} + \sin \dfrac{17\pi}{30}$.

(In what follows, the classical formulas used in the calculations will easily be recognized by the reader and won't be explicated.)

a) We have

$$A = \frac{1}{2}\left(1 - \cos\frac{2\pi}{9}\right) + \frac{1}{2}\left(1 - \cos\frac{4\pi}{9}\right) + \frac{1}{2}\left(1 - \cos\frac{8\pi}{9}\right) = \frac{3}{2} - \frac{A'}{2}$$

where

$$A' = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 2\cos\frac{\pi}{3}\cos\frac{\pi}{9} - \cos\frac{\pi}{9} = 0$$

and so $A = \frac{3}{2}$.

b) A frequently used trick will ease the calculation of B; instead of B alone, we consider $2B\cos\frac{\pi}{10}$:

$$2B\cos\frac{\pi}{10} = \sin\frac{2\pi}{15} - \sin\frac{\pi}{15} - \sin\frac{\pi}{3} - \sin\frac{2\pi}{15} - \sin\frac{7\pi}{15} - \sin\frac{4\pi}{15} + \sin\frac{2\pi}{3} + \sin\frac{7\pi}{15}$$

and so

$$2B\cos\frac{\pi}{10} = -\sin\frac{4\pi}{15} - \sin\frac{\pi}{15} = -2\sin\frac{\pi}{6}\cos\frac{\pi}{10} = -\cos\frac{\pi}{10},$$

which gives $B = -\dfrac{1}{2}$.

Here is a more elaborated example:

Prove that $$\frac{\sin\frac{5\pi}{18}}{\sin\frac{7\pi}{18}}+\frac{\sin\frac{7\pi}{18}}{\sin\frac{\pi}{18}}-\frac{\sin\frac{\pi}{18}}{\sin\frac{5\pi}{18}}=6.$$

Let us rewrite the left-hand side as $\frac{N}{D}$ where

$$D=\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}\quad\text{and}\quad N=\sin\frac{\pi}{18}\sin^2\frac{5\pi}{18}+\sin\frac{5\pi}{18}\sin^2\frac{7\pi}{18}-\sin\frac{7\pi}{18}\sin^2\frac{\pi}{18}.$$

We use the trick seen above to compute D:

$$2D\cos\frac{\pi}{18}=\sin\frac{\pi}{9}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}=\sin\frac{\pi}{9}\sin\frac{5\pi}{18}\cos\frac{\pi}{9}$$

(because $2\sin\frac{\pi}{18}\cos\frac{\pi}{18}=\sin\frac{\pi}{9}$ and $\frac{7\pi}{18}+\frac{\pi}{9}=\frac{\pi}{2}$). Continuing in the same vein, we obtain

$$4D\cos\frac{\pi}{18}=\sin\frac{2\pi}{9}\sin\frac{5\pi}{18}=\sin\frac{2\pi}{9}\cos\frac{2\pi}{9}$$

and finally $8D\cos\frac{\pi}{18}=\sin\frac{4\pi}{9}=\cos\frac{\pi}{18}$ so that $D=\frac{1}{8}$.

As for N, we transform products in sums and first get

$$N=\frac{1}{2}\sin\frac{5\pi}{18}\left(\cos\frac{2\pi}{9}-\cos\frac{\pi}{3}\right)+\frac{1}{2}\sin\frac{7\pi}{18}\left(\cos\frac{\pi}{9}-\cos\frac{2\pi}{3}\right)-\frac{1}{2}\sin\frac{\pi}{18}\left(\cos\frac{\pi}{3}-\cos\frac{4\pi}{9}\right)$$

and then

$$N=\frac{1}{4}\left(\sin\frac{7\pi}{18}-\sin\frac{5\pi}{18}-\sin\frac{\pi}{18}\right)+\frac{1}{4}\left(1+\sin\frac{\pi}{18}\right)+\frac{1}{4}\left(1+\sin\frac{5\pi}{18}\right)+\frac{1}{4}\left(1-\sin\frac{7\pi}{18}\right)$$
$$=\frac{3}{4}.$$

Thus $\frac{N}{D}=6$, as required.

With the help of additional tools

Complex numbers are closely related to trigonometry and this connection can often simplify proofs. To take a simple example, let us consider the following relation:

Show that

$$\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\cos\frac{6\pi}{5}+\cos\frac{8\pi}{5}=-1$$

and deduce the values of $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$.

The reader is invited to prove the result with trig formulas as in the previous paragraph; however, it seems preferable to use the complex number $w=\exp(2\pi i/5)$ as follows:

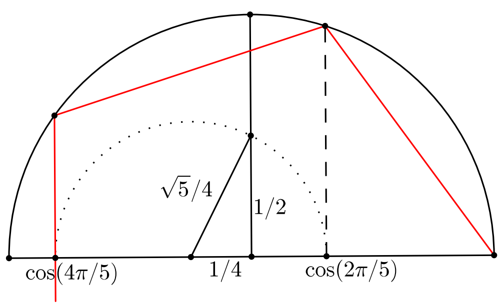
since $1+w+w^2+w^3+w^4 = \frac{1-w^5}{1-w} = 0$, we also have $\text{Re}(1+w+w^2+w^3+w^4) = 0$, which gives the desired result (because $\text{Re}(w^k) = \cos\frac{2k\pi}{5}$). The relation yields

$$-1 = 2\left(\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}\right) = 2\left(\cos\frac{2\pi}{5} + 2\cos^2\frac{2\pi}{5} - 1\right)$$

and solving the quadratic equation $4x^2 + 2x - 1 = 0$, we obtain

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}, \qquad \cos\frac{4\pi}{5} = \frac{-\sqrt{5}-1}{4}.$$

In passing, since $\frac{\sqrt{5}}{4}$ is the hypotenuse of a right-angled triangle whose other sides are $\frac{1}{4}$ and $\frac{1}{2}$, the results just obtained provide one of the simplest constructions of the regular pentagon.



In addition to complex numbers, polynomials often intervene in proofs. We illustrate this with two examples.

The first one is a problem set in the *American Math. Monthly* in 1999:

Prove that

$$\cos\frac{\pi}{7} = \frac{1}{6} + \frac{\sqrt{7}}{6}\left(\cos\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right) + \sqrt{3}\sin\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right)\right).$$

It seems wise to set $\theta = \arccos\frac{1}{2\sqrt{7}}$; with this notation, the proposed relation readily becomes

$$\cos\left(\frac{\pi-\theta}{3}\right) = \frac{6\cos\frac{\pi}{7} - 1}{2\sqrt{7}}.$$

Polynomials introduce themselves naturally if we recall the identity

$$\cos 3x = 4\cos^3 x - 3\cos x,$$

which shows that the number $u = \cos\left(\frac{\pi-\theta}{3}\right)$ satisfies

$$4u^3 - 3u = \cos(\pi - \theta) = -\frac{1}{2\sqrt{7}},$$

hence is a root of the polynomial $P(X) = 4X^3 - 3X + \frac{1}{2\sqrt{7}}$. Thus, we are led to seeking the roots of this polynomial. To this aim, a polynomial with $\cos\frac{\pi}{7}$ among its roots will be helpful. The role can be played by

$$Q(x) = x^3 - \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{8}$$

whose roots are $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$. Indeed, if we set

$$\begin{aligned} e_1 &= \cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7}, \\ e_2 &= \cos\frac{\pi}{7}\cos\frac{3\pi}{7} + \cos\frac{3\pi}{7}\cos\frac{5\pi}{7} + \cos\frac{5\pi}{7}\cos\frac{\pi}{7} \\ e_3 &= \cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7}, \end{aligned}$$

calculating $e_1 \sin\frac{\pi}{7}$ and $e_3 \sin\frac{\pi}{7}$ quickly provides $e_1 = \frac{1}{2}$ and $e_3 = -\frac{1}{8}$. In addition, transforming products into sums shows that $e_2 = -e_1$.

Now we have $Q(x) = \left(x - \frac{1}{6}\right)^3 - \frac{7}{12}\left(x - \frac{1}{6}\right) + \frac{7}{216}$ and so $Q\left(\frac{1}{6} + \frac{\sqrt{7}X}{3}\right) = \frac{7\sqrt{7}}{108} \cdot P(X)$. The roots of $P(X)$ are therefore

$$X_1 = \frac{6\cos\frac{\pi}{7} - 1}{2\sqrt{7}},\ X_2 = \frac{6\cos\frac{3\pi}{7} - 1}{2\sqrt{7}},\ X_3 = \frac{6\cos\frac{5\pi}{7} - 1}{2\sqrt{7}}.$$

It is easy to check that $X_2 < \frac{1}{2}, X_3 < 0$, while $u > \frac{1}{2}$ (since $0 < \frac{\pi-\theta}{3} < \frac{\pi}{3}$) and we can conclude that $u = X_1$, as desired.

Our second example, *Mathematics Magazine* problem 1562 posed in December 1998, mixes polynomials and complex calculations

$$\text{Prove that} \quad \tan\left(\frac{1}{4}\tan^{-1} 4\right) = 2\left(\cos\frac{6\pi}{17} + \cos\frac{10\pi}{17}\right).$$

With the help of the standard formulas, we readily obtain

$$\tan 4t = \frac{4\tan t - 4\tan^3 t}{1 - 6\tan^2 t + \tan^4 t}$$

and taking $t = \frac{1}{4}\tan^{-1} 4$ (so that $\tan 4t = 4$) we deduce that $x_0 = \tan\left(\frac{1}{4}\tan^{-1} 4\right)$ is a root of the polynomial $P(x) = x^4 + x^3 - 6x^2 - x + 1$.

Even better, noticing that $x_0 \in [0,1]$ and that P is a decreasing function on $[0,1]$, we see that x_0 is the only root of $P(x)$ in $[0,1]$. Thus, if $a = 2\left(\cos\frac{6\pi}{17} + \cos\frac{10\pi}{17}\right)$, we just have to show that $P(a) = 0$ and $a \in [0,1]$. The latter holds since

$$0 > \cos\frac{10\pi}{17} > \cos\frac{11\pi}{17} = -\cos\frac{6\pi}{17} > -\frac{1}{2}.$$

To prove the former, we introduce $u = \exp(2\pi i/17)$, so that the 17th roots of unity are the numbers u^k for $k = 0, 1, \ldots, 16$ and $a = u^3 + u^5 + u^{12} + u^{14}$. Conveniently, we also consider

$$b = u^2 + u^8 + u^9 + u^{15},$$
$$c = u^6 + u^7 + u^{10} + u^{11},$$
$$d = u + u^4 + u^{13} + u^{16}.$$

Using the equalities $u^{17} = 1$ and $-1 = \sum_{k=1}^{16} u^k = a + b + c + d$, and with a bit of algebra (and patience), we successively obtain:

$$\begin{aligned}
a^2 &= 4 + 2b + c,\\
ab &= 2a + c + d = a - b - 1,\\
ac &= a + b + c + d = -1,\\
a^3 &= 4a + 2ab + ac = 6a - 2b - 3,\\
a^4 &= 6a^2 - 2ab - 3a = 26 - 5a + 14b + 6c.
\end{aligned}$$

Lastly, a final calculation gives the expected result $a^4 + a^3 - 6a^2 - a + 1 = 0$.

It can be shown that the other roots of $P(x)$ are b, c, d. Relations analogous to the one of the statement can then be derived (left as an exercise to the reader).

The Quadratic Gauss Sums

Our last paragraph offers a few applications of the following beautiful theorem obtained by Gauss in 1805:

> **Theorem.** Let p be an odd prime number and $u = \exp(2\pi i/p)$. If p is of the form $4n + 3$ then $\sum_{k=1}^{p-1} \left(\frac{k}{p}\right) u^k = i\sqrt{p}$, while if p is of the form $4n + 1$, then $\sum_{k=1}^{p-1} \left(\frac{k}{p}\right) u^k = \sqrt{p}$.

[here $\left(\frac{k}{p}\right)$ is the Legendre symbol defined by $+1$ if k is a square modulo p and -1 otherwise.]

For a proof of this theorem, we refer the reader to [**1**] or [**2**].

As a first application, we consider Problem 218 proposed in 1983 in the *College Mathematics Journal*. (It is also a part of problem **2463** [1999: 366 ; 2000: 379]):

> Prove that $\tan\frac{3\pi}{11} + 4\sin\frac{2\pi}{11} = \sqrt{11}$.

We shall use two easy consequences of $e^{ix} = \cos x + i\sin x$:

$$4\sin x = -2i(e^{ix} - e^{-ix}) \quad \text{and} \quad \tan x = i\left(\frac{2}{1 + e^{2ix}} - 1\right).$$

On the one hand, the theorem with $p = 11$ gives

$$u - u^2 + u^3 + u^4 + u^5 - u^6 - u^7 - u^8 + u^9 - u^{10} = i\sqrt{11}.$$

On the other hand, we have

$$\frac{2}{1+u^3} = \frac{1+(u^3)^{11}}{1+u^3} = \sum_{k=0}^{10}(-u^3)^k = 1-u^3+u^6-u^9+u-u^4+u^7-u^{10}+u^2-u^5+u^8$$

and the formulas above lead to

$$\begin{aligned}&\tan\left(\frac{3\pi}{11}\right)+4\sin\left(\frac{2\pi}{11}\right)\\&= i(-u^3+u^6-u^9+u-u^4+u^7-u^{10}+u^2-u^5+u^8-2u+2u^{10})\\&= i(-i\sqrt{11}) = \sqrt{11}.\end{aligned}$$

Our last example, extracted from problem **3305** [2008 : 45,47 ; 2009 : 52], may seem of the same category as the previous one. However, some extra work will be needed, as can be seen in the variant of solution proposed here.

Prove that $\tan\frac{5\pi}{13}+4\sin\frac{2\pi}{13} = \sqrt{13+2\sqrt{13}}$.

We mimic the solution above, first applying the theorem with $p = 13$ to obtain $A - B = \sqrt{13}$ where

$$A = u+u^3+u^4+u^9+u^{10}+u^{12} \quad \text{and} \quad B = u^2+u^5+u^6+u^7+u^8+u^{11}.$$

In the same way as above, the calculations give

$$\tan\frac{5\pi}{13}+4\sin\frac{2\pi}{13} = i(C-D),$$

where

$$C = u^4+u^7+u^8+u^{10}+u^{11}+u^{12} \quad \text{and} \quad D = u+u^2+u^3+u^5+u^6+u^9.$$

Clearly, the connection is less direct than before! A look at the expected result prompts us to square and to show that

$$-(C-D)^2 = 13+2\sqrt{13},$$

that is,

$$4CD = 14+2\sqrt{13}$$

(note that $C+D=-1$ so that $4CD = (C+D)^2-(C-D)^2 = 1-(C-D)^2$).

Using $u^{13}=1$, the calculation of CD is lengthy but easy and yields

$$CD = 6+2(u^2+u^5+u^6+u^7+u^8+u^{11})+3(u+u^3+u^4+u^9+u^{10}+u^{12}) = 6+2B+3A.$$

Observing that $A+B=-1$ and $2A = (A+B)+(A-B) = -1+\sqrt{13}$, we conclude that

$$4CD = 24+8(A+B)+4A = 24-8-2+2\sqrt{13} = 14+2\sqrt{13}.$$

As usual, we propose a few exercises for the reader's practice.

Exercises

1. Evaluate

a) $\sin\dfrac{\pi}{14}\sin\dfrac{3\pi}{14}\sin\dfrac{5\pi}{14}$

b) $\tan\dfrac{6\pi}{7}+4\sin\dfrac{5\pi}{7}$.

2. Prove that

a) $\cos\dfrac{7\pi}{15}=4\sin\dfrac{2\pi}{15}\cos\dfrac{2\pi}{5}\cos\dfrac{13\pi}{30}$

b) $1+6\cos\dfrac{2\pi}{7}=2\sqrt{7}\cos\left(\dfrac{1}{3}\cos^{-1}\dfrac{1}{2\sqrt{7}}\right)$

(Problem 974 of the *College Mathematics Journal*).

References

[1] M. Guinot, *Une époque de transition, Lagrange et Legendre*, Aléas, 1996, p. 260

[2] K. Ireland, M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer, 1990, p. 70