CONTEST CORNER SOLUTIONS


CC291. The point $P$ is on the parabola $x^2 = 4y$. The tangent at $P$ meets the line $y = -1$ at the point $A$. For the point $F(0, 1)$, prove that $\angle AFP = 90^\circ$ for all positions of $P$, except $(0, 0)$.

*Originally Problem C3 from the 1998 Descartes Contest.*

We received 11 submissions, all correct. We present the solution by Dan Daniel.

Suppose the point $P$ has coordinates $(a, \frac{a^2}{4})$. For a parabola of the form $x^2 = 2py$, here $p = 2$, point $F$ is the focus and the line $y = -1$ is the directrix. Computing the tangent at $P$, we have $xa = 2(y + \frac{a^2}{4})$ and $y = -1$, which implies $x = \frac{a}{2} - \frac{a}{4}$, so point $A$ has coordinates $(\frac{a}{2} - \frac{a}{4}, -1)$. Then the slope of $AF$ is

$$s_1 = \frac{1 - (-1)}{0 - (\frac{a}{2} - \frac{a}{4})} = \frac{4a}{4 - a^2}$$

and the slope of $PF$ is

$$s_2 = \frac{\frac{a^2}{4} - 1}{a - 0} = \frac{a^2 - 4}{4a}.$$

Since $s_1 s_2 = -1$, $AF$ is perpendicular to $PF$ and so $\angle AFP = 90^\circ$.

CC292. Let $a$ be the length of a side and $b$ be the length of a diagonal in the regular pentagon $PQRST$ as shown.

\[ \frac{b}{a} - \frac{a}{b} = 1. \]

*Originally Problem D1 from the 1998 Descartes Contest.*
We received 12 solutions. We present the solution by Titu Zeonaru.

$PQRT$ is a cyclic quadrilateral, so by Ptolemy’s theorem

$$PR \cdot QT = PQ \cdot TR + PT \cdot QR;$$

in other words, $b^2 = ab + a^2$. Dividing both sides by $ab$ and rearranging gives us $b/a - a/b = 1$, as desired.

**CC293.** The transformation $T : (x, y) \mapsto (-\frac{1}{2}(3x - y), -\frac{1}{2}(x + y))$ is applied repeatedly to the point $P_0(3, 1)$, which produces a sequence of points $P_1, P_2, \ldots$. Show that the area of the convex quadrilateral defined by any four consecutive points is constant.

Originally Problem D2 from the 1998 Descartes Contest.

We received five submissions to this problem, all of which were correct and complete. We present the composite solution by Dan Daniel and Šefket Arslanagić (done independently), modified by the editor.

Given $P_n(x_n, y_n)$, it follows that

$$P_{n+2}(2x_n - y_n, x_n). \quad (1)$$

The slope of $P_nP_{n+2}$ is

$$\frac{y_n - x_n}{x_n - 2x_n + y_n} = 1.$$

Inductively, it follows that $P_0, P_2, P_4, \ldots$ are collinear along the line $y = x - 2$. Similarly, $P_1, P_3, P_5, \ldots$ are collinear points along the line $y = x + 2$. We note that these lines are parallel and that any convex quadrilateral defined by four consecutive points is a trapezoid with bases $P_nP_{n+2}$ and $P_{n+1}P_{n+3}$. Without loss of generality, assume that $P_nP_{n+2}$ lies along $y = x - 2$ and $P_{n+1}P_{n+3}$ lies along $y = x + 2$.

Given $P_n(x_n, x_n - 2)$, it follows from (1) that $P_{n+2}(x_n + 2, x_n)$. Therefore,

$$|P_nP_{n+2}| = \sqrt{2^2 + 2^2} = 2\sqrt{2}.$$**

The same methodology can be used to show that $|P_{n+1}P_{n+3}| = 2\sqrt{2}$. Therefore, the convex quadrilateral formed by the points $P_n, P_{n+1}, P_{n+2}, P_{n+3}$ has base lengths of $2\sqrt{2}$.

As $y = x - 2$ and $y = x + 2$ are parallel, we solve for the distance between the two lines with the points $(1, -1)$ and $(-1, 1)$, found on each line respectively:

$$\sqrt{(1+1)^2 + (-1-1)^2} = 2\sqrt{2}.$$**

Therefore, the height of the convex quadrilateral formed by the points $P_n, P_{n+1}, P_{n+2}, P_{n+3}$ is $2\sqrt{2}$.
The area of the convex quadrilateral formed by the points $P_n, P_{n+1}, P_{n+2}, P_{n+3}$ is therefore
\[
\frac{(2\sqrt{2} + 2\sqrt{2}) \cdot 2\sqrt{2}}{2} = 8.
\]

CC294.

a) Prove that $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$, where $0 < A < \pi/2$.

b) If $\sin 2A = 4/5$, find $\tan A$.

*Originally Problem B2 from the 1998 Descartes Contest.*

*We received 15 correct solutions. We present the solution by Iko Dimitrić.*

The statement, as printed, is false, due to a typo. Namely, if we let $A \rightarrow \pi/4$ then the left-hand side approaches 1, whereas the right-hand side approaches 0. The term $\tan 2A$ in the denominator should be corrected to $\tan^2 A$ for the statement to hold. Thus, we prove that
\[
\sin 2A = \frac{2 \tan A}{1 + \tan^2 A},
\]
where $0 < A < \pi/2$. Beginning with the right-hand side we have
\[
\frac{2 \tan A}{1 + \tan^2 A} = \frac{2 \tan A}{\sec^2 A} = 2 \cdot \frac{\sin A}{\cos A} \cdot \cos^2 A = 2 \sin A \cos A = \sin 2A,
\]
proving (a). Now let $x = \tan A$. Using the relation in part (a), we solve
\[
\frac{4}{5} = \frac{2x}{1 + x^2} \iff 4 + 4x^2 = 10x \iff 2(2x - 1)(x - 2) = 0,
\]
with two solutions $x = 2$ and $x = 1/2$. Thus $\tan A \in \{2, 1/2\}$.

CC295. In how many ways is it possible to choose four distinct integers from 1, 2, 3, 4, 5, 6 and 7, so that their sum is even?

*Originally Problem A5 from the 1998 Descartes Contest.*

*We received 10 submissions, all correct and complete. We present the solution by Kathleen E. Lewis.*

Since there are four odd numbers and three even numbers to choose from, one must either pick all four odd numbers or two odds and two evens. There is only one way to choose four odds, and \( \binom{4}{2} \cdot \binom{3}{2} = 18 \) ways to choose two odds and two evens. So altogether there are 19 ways to get four numbers whose sum is even.