OLYMPIAD SOLUTIONS


OC341. There are 30 teams in the NBA and every team plays 82 games in the year. Owners of the NBA teams want to divide all teams into Western and Eastern Conferences (not necessarily equally), such that the number of games between teams from different conferences is half the number of all games. Can they do it?

Originally 2016 AllRussian Olympiad Grade 11, Day 1, Problem 1.

We received only one solution. We present the solution by Mohammed Aassila.

We prove, by contradiction, that they cannot do it. We use graph theory. We construct a graph such that its vertices are the teams and we draw an edge between two teams if they have played with each other (it may be more than one edge between two teams). Now let A be the subgraph with Western teams and name B the subgraph with Eastern teams. We know that there are 615 edges between A and B. We know that in each graph, there are exactly an even number of vertices with odd degree, so we get that there are an even number of vertices in A with odd degree, and since the degree of each vertex is 82, each of these vertices send an odd number of edges to B, so there must be an even number of edges between A and B, contradiction.

OC342. Consider the second-degree polynomial $P(x) = 4x^2 + 12x - 3015$. Define the sequence of polynomials $P_1(x) = \frac{P(x)}{2016}$ and $P_{n+1}(x) = \frac{P(P_n(x))}{2016}$ for every integer $n \geq 1$.

a) Show that there exists a real number $r$ such that $P_n(r) < 0$ for every positive integer $n$.

b) For how many integers $m$ does $P_n(m) < 0$ hold for infinitely many positive integers $n$?

Originally 2016 Brazil National Olympiad Day 2, Problem 5.

We received 3 solutions. We present the solution by Mohammed Aassila.

Let $Q(x) = \frac{P(x)}{2016}$. Then, $P_1(x) = Q(x)$ and $P_{n+1}(x) = Q(P_n(x))$.

Define $P_0(x) = x$. We have

$$Q(x) = \frac{(2x + 3)^2 - 3024}{2016} = \frac{(x + \frac{3}{2})^2}{504} - \frac{3}{2}, \quad Q(x) + \frac{3}{2} = \left(\frac{x + \frac{3}{2}}{504}\right)^2.$$
Then, for every positive integer \( n \)

\[
P_n(x) + \frac{3}{2} = \left( P_{n-1}(x) + \frac{3}{2} \right)^2 = \ldots = \left( P_0(x) + \frac{3}{2} \right)^{2^n}.
\]

Hence,

\[
P_n(x) = 504 \cdot \left( \frac{x + \frac{3}{2}}{504} \right)^{2^n} - \frac{3}{2}.
\]

(a) \( P_n \left( -\frac{3}{2} \right) = -\frac{3}{2} < 0 \). Take \( r = -\frac{3}{2} \).

(b) \( P_n(m) < 0 \) if and only if \( \left( \frac{m + \frac{3}{2}}{504} \right)^{2^n} < \frac{1}{336} \). This holds for infinitely many positive integers \( n \) if and only if \( \left| \frac{m + \frac{3}{2}}{504} \right| < 1 \), i.e. if and only if \( -505.5 < m < 502.5 \). Therefore there are 1008 integers with this property.

**OC343.** Determine all pairs of positive integers \((a, n)\) with \( a \geq n \geq 2 \) for which \((a + 1)^n + a - 1\) is a power of 2.

*Originally 2016 Italian Mathematical Olympiad, Problem 4.*

*We received 3 solutions. We present the solution by Oliver Geupel.*

The pair \((a, n) = (4, 3)\) is a solution, and we prove that there are no other ones.

Let \((a, n)\) be any solution of the problem and let

\[
(a + 1)^n + a - 1 = 2^m. \tag{1}
\]

Since \( a \) divides \((a + 1)^n + a - 1\), it follows from (1) that \( a = 2^k \) for some positive integer \( k \). We have \( 2k \leq nk < m \), whence \( 2^m \equiv 0 \pmod{2^{2k}} \). By the binomial theorem,

\[
(a + 1)^n + a - 1 = \sum_{j=2}^{n} \binom{n}{j} 2^{jk} + (n + 1)2^k \equiv (n + 1)2^k \pmod{2^{2k}},
\]

Hence,

\[
(n + 1)2^k \equiv 0 \pmod{2^{2k}},
\]

which implies that \( a = 2^k \) is a divisor of \( n + 1 \). Since \( n \leq a \), we obtain \( n = 2^k - 1 \). Thus, \( n \geq 3 \).
We have $3k \leq nk < m$, whence $2^m \equiv 0 \pmod{2^{3k}}$. By the binomial theorem,
\[
(a + 1)^n + a - 1 = \binom{n}{2} 2^{2k} + (n + 1)2^k + \sum_{j=3}^{n} \binom{n}{j} 2^{jk} \\
\equiv \left(\frac{2^k - 1}{2}\right) 2^{2k} + 2^k \\
\equiv (2^{2k-1} - 2^k - 2^{k-1} + 2)2^{2k} \pmod{2^{3k}}.
\]
Hence,
\[
(2^{2k-1} - 2^k - 2^{k-1} + 2)2^{2k} \equiv 0 \pmod{2^{3k}}.
\]
Thus, $2 - 2^{k-1} \equiv 0 \pmod{2^k}$. It follows that $k = 2$. Consequently, $a = 4$ and $n = 3$.

**OC344.** Find all $a \in \mathbb{R}$ such that there exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying
\[
\begin{align*}
&\bullet \quad f(1) = 2016; \\
&\bullet \quad f(x + y + f(y)) = f(x) + ay \quad \forall x, y \in \mathbb{R}.
\end{align*}
\]
*Originally 2016 Vietnam National Olympiad Day 2, Problem 1.*

We received one solution. We present the solution by Mohammed Aassila.

Let $P(x, y)$ be the assertion : $f(x + y + f(y)) = f(x) + ay$. We have two cases.

*Case 1.* If $a \neq 0$, we immediately get that $f(x)$ is bijective. Then $P(x, 0)$ and injectivity imply that $f(0) = 0$. $P(0, y)$ implies $f(y + f(y)) = ay$ and so $x \to x + f(x)$ is surjective. $P(x, y)$ may then be written as
\[
f(x + (y + f(y))) = f(x) + f(y + f(y)).
\]
and so, since $x \to x + f(x)$ is surjective, $f(x)$ is additive. So
\[
f(f(1)) = f(2016) = 2016f(1) = 2016^2.
\]
and $P(0, 1)$ implies $a = 2016 + 2016^2$ and then $f(x) = 2016x$ is a solution.

*Case 2.* If $a = 0$, then $f(x) = 2016$ for all $x$ fits.

In conclusion the answer is $a \in \{0, 2016 \cdot 2017\}$.

**OC345.** Let $\triangle ABC$ be an acute triangle, and let $I_B$, $I_C$, and $O$ denote its $B$-excenter, $C$-excenter, and circumcenter, respectively. Points $E$ and $Y$ are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points $F$ and $Z$ are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$. Lines $\overline{I_B F}$ and $\overline{I_C E}$ meet at $P$. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

*Originally 2016 USAMO Day 1, Problem 3.*

We received 2 solutions and will present both of them.

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Solution 1, by Andrea Fanchini.

We use barycentric coordinates and the usual Conway’s notations with reference to the triangle $ABC$. Points $I_B$, $I_C$, $O$, $Y$, $Z$, $E$ and $F$ have coordinates

\[
I_B(a : -b : c), \quad I_C(a : b : -c), \quad O(a^2S_A : b^2S_B : c^2S_C)
\]

\[
Y(a : 0 : c), \quad Z(a : b : 0), \quad E(S_C : 0 : S_A), \quad F(S_B : S_A : 0)
\]

Let us find coordinates of point $P$. Lines $I_BF$ and $I_CE$ have equations

\[
I_BF : cS_Ax - cS_By - (aS_A + bS_B)z = 0, \quad I_CE : bS_Ax - (aS_A + cS_C)y - bS_Cz = 0.
\]

Then the point $P$ is

\[
P = I_BF \cap I_CE
\]

\[
= (a(aS_A + bS_B + cS_C) : b(aS_A + bS_B - cS_C) : c(aS_A - bS_B + cS_C)).
\]

We will next show that $PO$ and $YZ$ are perpendicular. Line $PO$ has equation

\[
PO : bc(cS_C - bS_B)x + ac(aS_A + cS_C)y - ab(aS_A + bS_B)z = 0
\]

and it has infinite point

\[
PO_\infty(-a(abc + bS_B + cS_C) : b(abc - aS_A + cS_C) : c(abc - aS_A + bS_B)).
\]

now line $YZ$ has equation

\[
YZ : bcx - acy - abz = 0
\]
and it has infinite perpendicular point

\[ YZ_{\infty} = (-a(abc + bS_B + cS_C) : b(abc - aS_A + cS_C) : c(abc - aS_A + bS_B)), \]

as needed.

Solution 2, by Oliver Geupel.

Let us apply trilinear coordinates relative to \( \triangle ABC \). For convenience, let us use shortcuts \( \alpha = \cos A, \beta = \cos B \) and \( \gamma = \cos C \). We have

\[ E = \gamma : 0 : \alpha, \quad F = \beta : 0 : \alpha, \quad I_B = 1 : -1 : 1, \quad I_C = 1 : 1 : -1, \quad O = \alpha : \beta : \gamma, \quad Y = 1 : 0 : 1, \quad Z = 1 : 1 : 0. \]

Since point \( P = p : q : r \) lies on the lines \( I_B F \) and \( I_C E \), we have

\[ \begin{vmatrix} 1 & -1 & 1 \\ \beta & \alpha & 0 \\ p & q & r \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ \gamma & 0 & \alpha \\ p & q & r \end{vmatrix} = 0, \]

Solving this system of two linear equations with repertoire methods, yields

\[ P = p : q : r = (\alpha + \beta + \gamma) : (\alpha + \beta - \gamma) : (\alpha - \beta + \gamma). \]

Equations for points \( x : y : z \) on the lines \( YZ \) and \( OP \) are

\[ -x + y + z = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ x & y & z \end{vmatrix} = 0 \]

and

\[ (\beta r - \gamma q)x + (\gamma p - \alpha r)y + (\alpha q - \beta p)z = \begin{vmatrix} \alpha & \beta & \gamma \\ p & q & r \\ x & y & z \end{vmatrix} = 0. \]

It is well-known that two lines \( \ell x + my + nz = 0 \) and \( \ell' x + m'y + n'z = 0 \) are perpendicular if

\[ T = \ell \ell' + mm' + nn' - (mn' + m'n)\alpha - (n\ell' + n'\ell)\beta - (\ell m' + \ell' m)\gamma \]

vanishes. With the equations of lines \( YZ \) and \( OP \) as above, \( T \) becomes

\[ -\beta(\alpha - \beta + \gamma) + \gamma(\alpha + \beta - \gamma) + \gamma(\alpha + \beta + \gamma) - \alpha(\alpha - \beta + \gamma) \]
\[ + \alpha(\alpha + \beta - \gamma) - \beta(\alpha + \beta + \gamma) \]
\[ - \alpha^2(\alpha + \beta - \gamma) + \alpha(\alpha + \beta + \gamma) - \alpha\gamma(\alpha + \beta + \gamma) + \alpha^2(\alpha - \beta + \gamma) \]
\[ + \beta^2(\alpha + \beta + \gamma) + \beta\gamma(\alpha + \beta - \gamma) + \alpha\beta(\alpha + \beta - \gamma) - \beta^2(\alpha + \beta + \gamma) \]
\[ + \gamma^2(\alpha + \beta + \gamma) - \alpha\gamma(\alpha - \beta + \gamma) - \beta\gamma(\alpha - \beta + \gamma) - \gamma^2(\alpha + \beta - \gamma) \]
\[ = 0. \]

Consequently, \( PO \perp YZ \).