Let us consider problem C3 from the 2017 COMC.

**Problem.** Let $XYZ$ be an acute-angled triangle. Let $s$ be the side-length of the square which has two adjacent vertices on side $YZ$, one vertex on side $XY$ and one vertex on side $XZ$. Let $h$ be the distance from $X$ to the side $YZ$ and $b$ the distance from $Y$ to $Z$.

(a) If the vertices have coordinates $X = (2, 4)$, $Y = (0, 0)$ and $Z = (4, 0)$, find $b$, $h$ and $s$.

(b) Given that $h = 3$ and $s = 2$, find $b$.

(c) If the area of the square is 2017, determine the minimum area of triangle $XYZ$.

**Discussion.**

As a rule, section C of the contest contains more advanced questions. However, items (a) and (b) are usually comparable to the easier questions from part A or B. Here in order to answer question (a), it could be very useful to draw a grid and place points $X, Y, Z$ in accordance with given coordinates as shown in the Figure 1. Then from the figure it is clear that the distance $b$ between $Y$ and $Z$ is 4, and the distance $h$ from $X$ to the side $YZ$ is 4 as well. Now, we may observe that points $(1, 2)$ and $(3, 2)$ lie on sides $XY$ and $XZ$ respectively and together with points $(1, 0)$ and $(3, 0)$ they define a square that satisfies the conditions of the problem. This square has side $s = 2$.

The lesson learned here is: a precise picture in a geometric problem could lead to a quick visual solution.
For future reference, let us call the vertices of the square $P, Q, R, S$ (see Figure 2). Alternatively to drawing a precise figure in order to find $s$ in (a), we note that $PQ$ is parallel to $YZ$, and thus triangle $XPQ$ is similar to $XYZ$. From these similar triangles we have the relation \[ \frac{s}{b} = \frac{h - s}{h}, \] that is, the ratio of the bases $\frac{PQ}{YZ} = \frac{s}{b}$ is the same as the ratio of corresponding altitudes \[ \frac{XN}{XM} = \frac{h - s}{h}. \] Since $h = b = 4$ in (a), we obtain $s = 2$. The above relation also allows us to answer (b) right away: because $h = 3$ and $s = 2$ in (b), we conclude that $b = \frac{sh}{h - s} = 6$.

The lesson learned here is: the same formula derived for the general situation could be useful for answering several particular questions.

Now we can attempt the more challenging question (c). We should take advantage of what we already learned from either Figure 1 or 2. This might lead us to different approaches: algebraic and geometric. An algebraic approach may start with noting that the area of the triangle is

\[ \frac{[XYZ]}{2} = \frac{bh}{2} = \frac{sh^2}{2(h - s)}. \]
Students who had already studied calculus may want to employ the derivative method in order to find the minimum of this expression. However, the knowledge of calculus is not necessary for solving this problem. For example, one possible approach is to look at the reciprocal of the area and to find the maximum of it instead. The advantage comes from the fact that the reciprocal is a quadratic function in the variable $\frac{1}{h}$, that is

$$\frac{2(h - s)}{h^2 s} = -2 \left( \frac{1}{h} \right)^2 + \frac{2}{s} \left( \frac{1}{h} \right),$$

so the maximum is achieved at $\frac{1}{h} = \frac{1}{2} \cdot \frac{1}{s}$ or equivalently, for $h = 2s$. Then $b = 2s$ and the area $[XYZ] = 2s^2 = 4034$.

Another interesting approach is as follows. Let $A = [XYZ]$, the area of the triangle $XYZ$. We already found above that $A = \frac{bh}{2} = \frac{sh^2}{2(h - s)}$. Rewrite this formula as a quadratic equation in $h$, namely,

$$sh^2 - 2Ah + 2As = 0.$$ 

In order for $h$ to be real the discriminant must be non-negative:

$$D = 4A^2 - 8As^2 = 4A(A - 2s^2) \geq 0.$$ 

This implies $A \leq 0$ or $A \geq 2s^2$. The former inequality is not possible. The latter shows that the minimum value of the area is $A = 2s^2 = 4034$.

The lesson learned here is: it is good to know more advanced techniques but it is even better if you can employ some more elementary smart ideas.

A pure geometric approach could be inspired by Figure 1, where we keep the grid, but remove the labels 1, 2, 3, 4, assuming that after an appropriate rescaling of Figure 1 the area of $PQRS$ is 2017.

![Figure 3: The area of $XYZ$ is twice the area of $PQRS$.](image)
Now we simply count the squares of the grid: the square $PQRS$ consists of 4 squares of the grid, the area of $XPQ$ is 2 squares of the grid and each of $YPR$ and $ZQS$ has the area of one square of the grid. We conclude that the area of triangle $XYZ$ is exactly twice the area of the square $PQRS$, and thus it is 4034. We will demonstrate now that this is the smallest possible area for $XYZ$.

First we observe that if we keep the square $PQRS$ in the original place and move the vertex $X$ strictly left (or right) in the position $X_1$ (Figure 4), the area of the resulting triangle $X_1Y_1Z_1$ is the same as the area of $XYZ$ because their altitudes $XM = X_1M_1$ and the length $YZ$ is the same as $Y_1Z_1$ (since $YY_1 = XX_1 = ZZ_1$ from congruent triangles $PYY_1 \cong PXX_1$ and $QZZ_1 \cong QXX_1$).

Second, if we keep the square $PQRS$ in the original place and move the vertex $X$ strictly up (or down) to the position of $X_1$, then vertices $Y$ and $Z$ will move to the positions of $Y_1$ and $Z_1$ respectively (Figure 5). However now the area of the

Figure 4: the area of $X_1Y_1Z_1$ is equal to the area of $XYZ$, which is twice the area of $PQRS$.

Figure 5: the area of $XYZ$ is less than the area of $X_1Y_1Z_1$. 

Copyright © Canadian Mathematical Society, 2018
triangle $X_1Y_1Z_1$ will be bigger than the area of $XYZ$ by the value of the area of $UVX_1$, because the following triangles are pairwise congruent: $QVX \cong QZ_1Z$ and $PUX \cong PY_1Y$.

Finally, if $X$ moves in any direction, this could be viewed as a composition of two movements: strictly left or right, which does not change the area of $XYZ$, and then strictly up or down, which increases the area, thus the minimum area of $XYZ$ is 4034.

The lesson we learned here is: a precise picture could provide an quick idea for answering some questions, but then we need to justify our hypothesis in full generality.

In conclusion, many problems, especially geometrical ones often have several different approaches to find the answer. We have demonstrated some of them and you are welcome to explore your own! In addition, you may challenge yourself with related questions, for example: how to construct a square $PQRS$, given a particular triangle $YXZ$?

---

**Fun with clocks**

The 12 dots on the circumference are equally spaced and the only point used inside the circle is its centre. What fraction of each circle is coloured?

*Puzzle by Catriona Shearer ([https://twitter.com/Cshearer41](https://twitter.com/Cshearer41)).*