THE CONTEST CORNER

No. 68
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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d’un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n’importe quel problème. S’il vous plaît vous référez aux règles de soumission à l’endos de la couverture ou en ligne.

Pour faciliter l’examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 1er mars 2019.

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CC336. Soit la matrice de Pascal \(n \times n\) définie de la façon suivante : \(a_{i,j} = a_{i1} = 1,\) et \(a_{i,j} = a_{i-1,j} + a_{i,j-1}\) pour tout \(i, j > 1.\) Par exemple, la matrice de Pascal \(3 \times 3\) est donnée par

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
1 & 3 & 6
\end{bmatrix}.
\]

Montrer que toute matrice de Pascal est inversible.

CC337. Soit \(P(x)\) et \(Q(x)\) des polynômes à coefficients réels. Trouver les conditions nécessaires et suffisantes sur \(N\) pour garantir que si le polynôme \(P(Q(x))\) est de degré \(N,\) il existe un nombre réel \(x\) tel que \(P(x) = Q(x)\).

CC338. Trouver (avec preuve) toutes les solutions entières \((x, y)\) à \(x^2 - xy + 2017y = 0.\)

CC339. Un dodécaèdre rhombique (granatoèdre) a douze faces rhombiques congruentes ; chaque sommet a soit quatre petits angles ou trois grands angles qui s’y rencontrent. Si la mesure du côté est de 1, trouver le volume sous la forme \(\frac{p + \sqrt{q}}{r},\) où \(p, q,\) et \(r\) sont des nombres naturels et \(r\) n’a ni de facteur commun avec \(p\) ni avec \(q.\)
CC340. Soit $S$ l’ensemble des nombres naturels qui divisent $2018^{2018}$. De combien de façons peut-on sélectionner trois nombres \{x, y, z\} (pas nécessairement distincts, mais dont l’ordre est sans importance) de $S$ tels que $y = \sqrt{xyz}$ ?

CC336. Define the $n \times n$ Pascal matrix as follows: $a_{1j} = a_{i1} = 1$, while $a_{ij} = a_{i-1,j} + a_{i,j-1}$ for $i, j > 1$. So, for instance, the $3 \times 3$ Pascal matrix is

$$
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
1 & 3 & 6
\end{bmatrix}.
$$

Show that every Pascal matrix is invertible.

CC337. Suppose $P(x)$ and $Q(x)$ are polynomials with real coefficients. Find necessary and sufficient conditions on $N$ to guarantee that if the polynomial $P(Q(x))$ has degree $N$, there exists real $x$ with $P(x) = Q(x)$.

CC338. Find (with proof) all integer solutions $(x, y)$ to $x^2 - xy + 2017y = 0$.

CC339. A rhombic dodecahedron has twelve congruent rhombic faces; each vertex has either four small angles or three large angles meeting there. If the edge length is 1, find the volume in the form $\frac{p+\sqrt{q}}{r}$, where $p, q$, and $r$ are natural numbers and $r$ has no factor in common with $p$ or $q$.

CC340. Let $S$ be the set of natural numbers dividing $2018^{2018}$. In how many ways can one select three numbers \{x, y, z\} (not necessarily distinct, but order being irrelevant) from $S$ so that $y = \sqrt{xyz}$?
CONTEST CORNER
SOLUTIONS


CC286. The function \(f(n) = an + b\), where \(a\) and \(b\) are integers, is such that for every integer \(n\), \(f(3n+1), f(3n)+1\) and \(3f(n)+1\) are three consecutive integers in some order. Determine all such \(f(n)\).

Problem 1 of the 2008 Alberta High School Mathematics Competition, Part II.

We received five correct solutions. We present the solution of Ivko Dimitrić.

Since
\[
\begin{align*}
    f(3n+1) &= 3an + (a + b), \\
    f(3n) + 1 &= 3an + (b + 1), \\
    3f(n) + 1 &= 3an + (3b + 1)
\end{align*}
\]

are consecutive integers in some order, then, upon subtracting the common term \(3an\) from each, it follows that \(a+b, b+1\) and \(3b+1\) are consecutive integers, whereas \(b\) cannot be 0 since these numbers are different. Then \(|(3b+1)-(b+1)| = |2b| \geq 2\).

We see that \(3b+1\) and \(b+1\) cannot be consecutive integers, so \(a+b\) has to be between them and the following are the possibilities :

1. \(b+1 < a+b < 3b+1\) if \(b > 0\) or
2. \(3b+1 < a+b < b+1\) if \(b < 0\).

In either case these are three consecutive integers.

In case (1), we have \(a+b = b+2 = 3b\), giving \(b = 1\), \(a = 2\), so that the function is \(f(n) = 2n+1\). In case (2), we have \(a+b = 3b+2 = b\), with solution \(a = 0\), \(b = -1\), so the function is \(f(n) = 0 \cdot n + (-1) = -1\). These two are the only such functions.

CC287. In a contest, no student solved all problems. Each problem was solved by exactly three students and each pair of problems was solved by exactly one student. What is the maximum number of problems in this contest?

Problem 2 of the 2008 Alberta High School Mathematics Competition, Part II.

We received only one solution and it was incorrect. We present a solution created by this question’s editor, Allen O’Hara.

Suppose there are \(n\) problems on the contest, \(q_1, q_2, \ldots, q_n\).

Each question is solved by 3 different students. Consider the first problem on the test, question \(q_1\). We’ll say it was solved by students \(s_1, s_2, s_3\). What other questions might they have solved? Since every pair of questions has been answered
by exactly one student, we see that the set \( q_2, q_3, \ldots, q_n \) is partitioned into 3 subsets, \( A_1, A_2, \) and \( A_3 \). At most one can be empty as then we’d have one student answer all the questions which is prohibited. We may assume that \( \|A_1\| \geq \|A_2\| \geq \|A_3\| \), with possibly \( \|A_3\| = 0 \).

Suppose \( \|A_1\| > 2 \). Then without loss of generality we can say \( q_2, q_3, q_4 \in A_1 \) and \( q_5 \in A_2 \). But then the pairs \( (q_2, q_5), (q_3, q_5), (q_4, q_5) \) must be solved by some students. This gives rise to three more distinct students, as none of the \( q_2, q_3, q_4 \) can be solved by the same student, since student \( s_1 \) has already solved those pairwise groupings. This means that we have at least four students who have solved \( q_5 \), a contradiction.

So we have an upper bound, \( 2 \geq \|A_1\| \geq \|A_2\| \geq \|A_3\| \). Since these sets partition \( \{q_2, q_3, \ldots, q_n\} \) this gives \( 6 \geq n - 1 \) and so \( n \) is at most 7. Attempting to create a solution with \( n = 7 \) will give rise to the following possible situation, and so the final answer of 7 problems in the contest is realized.

Student \( s_1 \) solves questions \( q_1, q_2, \) and \( q_3 \).
Student \( s_2 \) solves questions \( q_1, q_4, \) and \( q_5 \).
Student \( s_3 \) solves questions \( q_1, q_6, \) and \( q_7 \).
Student \( s_4 \) solves questions \( q_2, q_4, \) and \( q_6 \).
Student \( s_5 \) solves questions \( q_2, q_5, \) and \( q_7 \).
Student \( s_6 \) solves questions \( q_3, q_4, \) and \( q_7 \).
Student \( s_7 \) solves questions \( q_3, q_5, \) and \( q_6 \).

Editor’s comment. It is interesting to note the connection between this solution and Steiner triples.

**CC288.** The lengths of the sides of triangle \( ABC \) are consecutive positive integers. \( D \) is the midpoint of \( BC \) and \( AD \) is perpendicular to the bisector of angle \( C \). Determine the product of the lengths of the three sides.

*Problem 15 of the 2011 Alberta High School Mathematics Competition, Part I.*

We received six correct solutions to this problem. We present the joint solution of Miguel Amengual Covas and Titu Zvonaru, modified by the editor.
Let the internal bisector of angle \( C \) meets \( AD \) at \( M \). Since \( CM \perp AD \), triangles \( AMC \) and \( DMC \) are similar.

\( \triangle AMC \) and \( \triangle DMC \) are congruent since both have a common side \( CA = CD \). Thus \( CA = \frac{1}{2} BC \). As \( CA < BC \), we examine the following three cases:

(i) If \( CA = n \) and \( BC = n + 1 \) then \( n = \frac{1}{2} (n + 1) \), which implies that \( n = 1 \).

In this case, \( \triangle ACB \) has side lengths 1, 2, 3. However, this case violates the triangle inequality as \( AB < AC + BC \Rightarrow 3 < 1 + 2 \) does not hold.

(ii) If \( CA = n \) and \( BC = n + 2 \), then \( n = \frac{1}{2} (n + 2) \), which implies that \( n = 2 \).

In this case, \( \triangle ACB \) has side lengths 2, 4, 3.

(iii) If \( CA = n + 1 \) and \( BC = n + 2 \), then \( n + 1 = \frac{1}{2} (n + 2) \), which implies that \( n = 0 \).

Thus the solution to this problem is \( 4 \cdot 3 \cdot 2 = 24 \).

**CC289.** A positive integer is said to be *special* if it can be written as the sum of the square of an integer and a prime number. For example, 101 is special because 101 = 64 + 37. Here 64 is the square of 8 and 37 is a prime number.

a) Show that there are infinitely many positive integers which are special.

b) Show that there are infinitely many positive integers which are not special.

*Problem 3 of the 2012 Alberta High School Mathematics Competition, Part II.*

*We received six correct solutions. We present the solution of David Manes.*

For each positive integer \( n \), let \( p_n \) be the \( n^{th} \) prime. Then

\[ p_n + 1 = 1^2 + p_n = 1 + p_n. \]

Hence, \( p_n + 1 \) is special. Since there are infinitely many primes, the result in part a) follows.

For part b), we will show that for each positive integer \( n \), the integer \( (3n + 5)^2 \) cannot be written as the sum of a square and a prime. Assume the contrary that

\[ (3n + 5)^2 = m^2 + p \]

for some positive integer \( m \) and prime \( p \). Then

\[ p = (3n + 5)^2 - m^2 = (3n + 5 - m)(3n + 5 + m). \]

If \( 3n + 5 - m > 1 \), then \( p \) is not a prime since it is written as the product of two integers greater than 1. Therefore, \( 3n + 5 - m = 1 \) in which case \( m = 3n + 4 \) and

\[ p = 3n + 5 + m = 6n + 9 = 3(2n + 3) \]

and again \( p \) is composite since \( 2n + 3 \geq 5 \). Hence, the result in part b) follows.

*Cruz Mathematicorum, Vol. 44(8), October 2018*
CC290. Randy plots a point \( A \). Then he starts drawing some rays starting at \( A \), so that all the angles he gets are integral multiples of \( 10^\circ \). What is the largest number of rays he can draw so that all the angles at \( A \) between the rays are unequal, including all angles between non-adjacent rays?

*Problem 3 of the 2013 Alberta High School Mathematics Competition, Part II.*

We received four correct solutions. We present the solution by Ivko Dimitrić.

If seven or more rays are drawn then there are at least \( \binom{7}{2} = \frac{7 	imes 6}{2} = 21 \) different pairs of rays, determining 21 convex angles (angles between \( 10^\circ \) and \( 180^\circ \)) between them. Their angle measures cannot be all different, since they are integral multiples of \( 10^\circ \) and there are only 18 such multiples between \( 10^\circ \) and \( 180^\circ \) inclusive.

On the other hand, 6 rays can be drawn forming \( \binom{6}{2} = 15 \) different angles whose measures are integral multiples of \( 10^\circ \).

One such arrangement is obtained when the rays are drawn so that the angle measures of successive angles formed by pairs of adjacent rays in cyclic order are \( 30^\circ, 10^\circ, 70^\circ, 50^\circ, 140^\circ, 60^\circ \). In this arrangement all the angles between \( 10^\circ \) and \( 180^\circ \) except \( 20^\circ, 150^\circ, 180^\circ \) are obtained exactly once by a pair of rays.