Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 1er mars 2019.

La rédaction souhaite remercier Valérie Lapointe, Carignan, QC, d’avoir traduit les problèmes.

**OC396.** Montrer qu'il existe une infinité d'entiers positifs $m$ tels que le nombre de facteurs premiers impairs de $m(m + 3)$ est un multiple de 3.

**OC397.** Soit le triangle $ABC$ tel que $\angle A = 45^\circ$ et $AM$ est une médiane. Soit le segment $b$ symétrique au segment $AM$ par rapport à la hauteur $BB_1$ et le segment $c$ symétrique au segment $AM$ par rapport à la hauteur $CC_1$. Les segments $b$ et $c$ s’interceptent au point $X$. Montrer que $AX = BC$.

**OC398.** Le détective Nero Wolfe enquête sur un crime. Il y a 80 personnes d’intérêt dans cette affaire. Parmi ces personnes, on retrouve le criminel et le témoin du crime (mais on ne sait pas qui est qui). Tous les jours, le détective peut interroger une ou plusieurs des ces 80 personnes. Si parmi les personnes invitées, le témoin est appelé et que le criminel ne l'est pas, le témoin pourra dire qui est le criminel. Est-ce que le détective peut résoudre ce crime en 12 jours ?

**OC399.** On dit qu’une fonction $f : \mathbb{R} \rightarrow \mathbb{R}$ a la propriété $\mathcal{P}$ si pour toute suite de nombres réels $(x_n)_{n \geq 1}$ telle que la suite $(f(x_n))_{n \geq 1}$ converge, alors la suite $(x_n)_{n \geq 1}$ converge. Montrer qu’une fonction surjective avec la propriété $\mathcal{P}$ est continue.

**OC400.** Soit $G$ un groupe fini ayant la propriété suivante : pour tout automorphisme $f$ de $G$, il existe un nombre naturel $m$ tel que $f(x) = x^m$ pour tout $x \in G$. Montrer que $G$ est abélien.
**OC396.** Prove that there are infinitely many positive integers \( m \) such that the number of odd distinct prime factors of \( m(m + 3) \) is a multiple of 3.

**OC397.** In a triangle \( ABC \) with \( \angle A = 45^\circ \), draw the median \( AM \). The line \( b \) is symmetrical to the line \( AM \) with respect to the altitude \( BB_1 \) and the line \( c \) is symmetrical to \( AM \) with respect to the altitude \( CC_1 \). The lines \( b \) and \( c \) intersect at the point \( X \). Prove that \( AX = BC \).

**OC398.** Detective Nero Wolfe is investigating a crime. There are 80 people involved in this case, among them one is the criminal and another is a witness of the crime (but it is not known who is who). Every day, the detective can invite one or more of these 80 people for an interview; if among the invited there is the witness, but there is no criminal, then the witness will tell who the criminal is. Can the detective solve the case in 12 days?

**OC399.** We say that a function \( f : \mathbb{R} \to \mathbb{R} \) has the property \( \mathcal{P} \) if for any sequence of real numbers \( (x_n)_{n \geq 1} \) such that the sequence \( (f(x_n))_{n \geq 1} \) converges, then also the sequence \( (x_n)_{n \geq 1} \) converges. Prove that a surjective function with property \( \mathcal{P} \) is continuous.

**OC400.** Let \( G \) be a finite group having the following property: for any automorphism \( f \) of \( G \), there exists a natural number \( m \) such that \( f(x) = x^m \) for all \( x \in G \). Prove that \( G \) is abelian.
OLYMPIAD SOLUTIONS


OC336. Find all functions \( f : \mathbb{N} \to \mathbb{N} \) such that for all \( a, b \in \mathbb{N} \), we have
\[
(f(a) + b)f(a + f(b)) = (a + f(b))^2.
\]

Originally Problem 2 from the 2016 Iranian Mathematical Olympiad (3rd Round).
Michel Bataille notified us that OC336 appeared before in our journal as problem 4040 of 41(4) and solved in 42(4). OC336 received two solutions, both of which were correct, complete and different than previously published solutions. We present the solution by Steven Chow.

We assume that both domain and codomain of the function \( f \) are the set of all integers greater than or equal to 1. The equation that defines the function \( f \) is
\[
(f(x) + y) f(x + f(y)) = (x + f(y))^2. 
\]

Take \( x = a \) and \( y = a \) in (1) to obtain \( (f(a) + a) f(a + f(a)) = (a + f(a))^2 \).
Since \( a + f(a) \geq 1 + 1 \), it follows that for all integers \( a \geq 1 \)
\[
f(a + f(a)) = a + f(a). 
\]

Using mathematical induction, we shall prove that the statement
\[
P(x) : f(xa) = xf(a) \text{ for all integers } a \geq 1
\]
is true for all integers \( x \geq 1 \). It is obvious that \( P(1) \) is true. Assume that for some integer \( k \geq 1 \), \( P(k) \) holds, equivalently \( f(ka) = kf(a) \) for all integers \( a \geq 1 \).

We evaluate (1) at \( x = ka \) and \( y = a + f(a) \):
\[
(f(ka) + a + f(a)) f(ka + f(a + f(a))) = (ka + f(a + f(a)))^2. 
\]

Because of (2) and \( f(ka) = kf(a) \), (3) writes as
\[
(a + (k + 1)f(a)) f((k + 1)a + f(a)) = ((k + 1)a + f(a))^2. 
\]

We evaluate (1) at \( x = (k + 1)a \) and \( y = a \):
\[
(f((k + 1)a) + a) f((k + 1)a + f(a)) = ((k + 1)a + f(a))^2 
\]

The right sides of (4) and (5) are the same, implying that the left sides of (4) and (5) are equal; and taking into account that \( f((k + 1)a + f(a)) \geq 1 \), it follows that \( a + (k + 1)f(a) = f((k + 1)a + a) \), or equivalently \( (k + 1)f(a) = f((k + 1)a) \) for any integer \( a \geq 1 \). Hence \( P(k + 1) \) is established; end of induction.

Crux Mathematicorum, Vol. 44(8), October 2018
Therefore for all integers \( x \geq 1 \) and \( a \geq 1 \), \( f(xa) = xf(a) \). Since \( f(x) = f(1)x \) and \( a + f(a) = f(a + f(a)) = f(1)(a + f(a)) \) it follows that \( f(1) = 1 \). Hence for all integers \( x \geq 1 \), \( f(x) = x \). Indeed, this function satisfies the condition (1).

In conclusion, the only solution of (1) is the identity function, \( f(x) = x \) for all integers \( x \geq 1 \).

**OC337.** Find all polynomials \( P(x) \) with integer coefficients such that

\[
P(P(n) + n)
\]

is a prime number for infinitely many integers \( n \).

*Originally Problem 3 from the 2016 Canadian Mathematical Olympiad.*

We received three solutions, all of which were correct and complete. We present the solution by Jose Luis Diaz-Barrero.

Let \( P(x) \) be a polynomial with integer coefficients of degree \( m \). That is,

\[
P(x) = \sum_{k=0}^{m} a_k x^k
\]

with \( a_k \in \mathbb{Z} \) for \( 0 \leq k \leq m \). Next we consider two cases.

First, if \( P(x) \) is a constant polynomial equal to \( p \), then \( P(P(n) + n) = p \) for any integer \( n \). Hence \( P(P(n) + n) \) is a prime number for infinitely many integers \( n \) if and only if \( p \) is a prime number.

Second, assume that \( P(x) \) is not a constant polynomial, hence its degree \( \deg(P) \geq 1 \). We prove that there exists a polynomial \( Q(x) \) such that \( P(P(x) + x) = P(x)Q(x) \) and \( \deg(Q) = (\deg(P))^2 - \deg(P) \).

\[
P(P(x) + x) - P(x) = \sum_{k=0}^{m} a_k (P(x) + x)^k - \sum_{k=0}^{m} a_k x^k
\]

\[
= \sum_{k=1}^{m} a_k ((P(x) + x)^k - x^k)
\]

\[
= \sum_{k=1}^{m} a_k P(x) \sum_{l=0}^{k-1} P(x)^l x^{k-1-l}
\]

\[
= P(x) \sum_{k=1}^{m} \sum_{l=0}^{k-1} a_k P(x)^l x^{k-1-l} = P(x)Q(x),
\]

with \( Q(x) = \sum_{k=1}^{m} \sum_{l=0}^{k-1} a_k P(x)^l x^{k-1-l} \). In addition

\[
\deg(Q) = \deg(P(P(x) + x)) - \deg(P) = (\deg(P))^2 - \deg(P).
\]

Since \( P(P(n) + n) = P(n)Q(n) \) is prime for infinitely many integers \( n \), it follows that either \( P(n) = \pm 1 \) for infinitely many integers \( n \) or \( Q(n) = \pm 1 \) for infinitely
many integers \( n \). Since \( P(x) \) is a polynomial of degree \( m \geq 1 \), \( P(n) = \pm 1 \) cannot hold for infinitely many \( n \). So, we have for infinitely many integers \( n \), \( Q(n) = \pm 1 \), equivalently \( \deg(Q) = 0 \). Therefore \((\deg(P))^2 - \deg(P) = 0, \deg(P) = 1\).

Let \( P(x) = a_0 + a_1 x \) for some integers \( a_0 \) and \( a_1 \). Since

\[
P(P(x) + x) = a_0 + a_1(a_0 + a_1 x + x) = P(x)(1 + a_1)
\]

it follows that \( Q(x) = 1 + a_1 = \pm 1 \). The case \( 1 + a_1 = 1 \) leads to \( a_1 = 0 \) which is impossible since we assumed \( \deg(P) \geq 1 \). The case \( 1 + a_1 = -1 \) leads to \( a_1 = -2 \), \( P(x) = a_0 - 2x \), and \( P(P(x) + x) = a_0 - 2x \). Lastly \( P(P(n) + n) = a_0 - 2n \) is prime for infinitely many integers \( n \) if and only if \( a_0 \) is odd.

Finally, we conclude that constant polynomial \( P(x) = p \) with \( p \) a prime number or \( P(x) = a_0 - 2x \) with \( a_0 \) an odd integer are the polynomials \( P(x) \) that fulfill the conditions of the statement.

**OC338.** Let \( \Gamma \) be the excircle of triangle \( ABC \) opposite to the vertex \( A \) (namely, the circle tangent to \( BC \) and to the extensions of the sides \( AB \) and \( AC \) from the points \( B \) and \( C \)). Let \( D \) be the center of \( \Gamma \) and \( E, F \), respectively, the points in which \( \Gamma \) touches the extensions of \( AB \) and \( AC \). Let \( J \) be the intersection between the segments \( BD \) and \( EF \). Prove that \( \angle CJB \) is a right angle.

*Originally Problem 3 from the 2016 Italian Mathematical Olympiad.*

We received six solutions, all of which were correct and complete. Four solutions were geometrical, one solution used complex numbers, and one solution used barycentric coordinates. We present 2 solutions.

**Solution 1, by Mohammed Aassila.**

Denote

\[
\angle A = \angle BAC, \quad \angle B = \angle CBA, \quad \text{and} \quad \angle C = \angle ACB.
\]

Since the angles of a triangle add up to 180°, we obtain for triangle \( BDC \):

\[
\angle BDC = 180^\circ - \angle DCB - \angle CBD = 180^\circ - \frac{180^\circ - \angle C}{2} - \frac{180^\circ - \angle B}{2} = \frac{180^\circ - \angle A}{2}.
\]

Since triangle \( AEF \) is isosceles, we obtain:

\[
\angle JFC = \frac{180^\circ - \angle A}{2}.
\]

So the points \( J, C, F, \) and \( D \) lie on the same circle, hence

\[
\angle CJD = 180^\circ - \angle DFC = 90^\circ,
\]

and

\[
\angle CJB = 180^\circ - \angle CJD = 90^\circ.
\]
Solution 2, by Oliver Geupel.

Let $K$ be the point of tangency of the line $BC$ to $\Gamma$. Let $LL'$ be the diameter of $\Gamma$ with the property that the points $B$, $L$, and $L'$ are collinear. Let $M$ be the midpoint of $BC$. Put the picture onto a complex plane such that $\Gamma$ is the unit circle, and identify each point with the corresponding complex number.

It is well-known that the intersection of tangents from tangency points $X$ and $Y$ to the unit circle is the point $2XY/(X + Y)$. Hence,

$$B = \frac{2EK}{E + K}, \quad C = \frac{2FK}{F + K}, \quad M = \frac{B + C}{2} = \frac{K(2EF + EK + FK)}{(E + K)(F + K)}.
$$

Then

$$|B - M|^2 = \frac{K^2(E - F)}{(E + K)(F + K)} \cdot \frac{\frac{1}{E^2} - \frac{1}{F^2}}{\left(\frac{1}{E} + \frac{1}{K}\right)\left(\frac{1}{F} + \frac{1}{K}\right)} = -\frac{K^2(E - F)^2}{(E + K)^2(F + K)^2}.
$$

Since $\angle KDL = \angle LDE$, we have $L^2 = EK$. It is a well-known and easily verifiable fact that the intersection of chords $\{X, Y\}$ and $\{Z, U\}$ of the unit circle is the point

$$(XY(Z + U) - ZU(X + Y))/(XY - ZU).
$$

We obtain for the intersection $J$ of chords $\{E, F\}$ and $\{L, -L\}$:

$$J = \frac{L^2(E + F)}{EF + L^2} = \frac{EK(E + F)}{EF + EK} = \frac{K(E + F)}{F + K^2},
$$

and

$$|J - M|^2 = \frac{EK(E - F)}{(E + K)(F + K)} \cdot \frac{\frac{1}{E^2} - \frac{1}{F^2}}{\left(\frac{1}{E} + \frac{1}{K}\right)\left(\frac{1}{F} + \frac{1}{K}\right)} = -\frac{K^2(E - F)^2}{(E + K)^2(F + K)^2} = |B - M|^2.
$$
Thus, $J$ lies on the circle with center $M$ and radius $BM$. This circle has $CB$ as diameter, therefore $\angle CJB = 90^\circ$.


**OC339.** Let $n$ be any positive integer. Prove that

$$\sum_{i=1}^{n} \frac{1}{(i^2 + i)^{3/4}} > 2 - \frac{2}{\sqrt{n + 1}}.$$

*Originally Problem 3 from the 2016 Philippines Mathematical Olympiad.*

We received 8 solutions, all of which were complete and correct. 7 solutions used AM-GM inequality, of which 4 solutions were completed via mathematical induction. One solution used the monotonicity of a function. We present a solution that was independently proposed by Mohammad Aassila, Steven Chow, and Oliver Geupel.

For all integers $j \geq 1$,

$$\frac{1}{\sqrt{j}} > \frac{1}{\sqrt{j+1}} > 0,$$

so from the AM - GM inequality,

$$\frac{1}{\sqrt{j}} + \frac{1}{\sqrt{j+1}} > \frac{2}{(j^2 + j)^{1/2}}.$$

In the above inequality, multiply both sides by $1/ (j^2 + j)^{3/2}$ to obtain

$$\frac{1}{(j^2 + j)^{3/2}} \left( \frac{2}{\sqrt{j}} + \frac{2}{\sqrt{j+1}} \right) > \frac{4}{j^2 + j}$$

$$= \frac{4}{j} - \frac{4}{j + 1}$$

$$= \left( \frac{2}{\sqrt{j}} + \frac{2}{\sqrt{j+1}} \right) \left( \frac{2}{\sqrt{j}} - \frac{2}{\sqrt{j+1}} \right),$$

so

$$\frac{1}{(j^2 + j)^{3/2}} > \frac{2}{\sqrt{j}} - \frac{2}{\sqrt{j+1}}.$$ 

Therefore

$$\sum_{i=1}^{n} \frac{1}{(i^2 + i)^{3/2}} > \sum_{i=1}^{n} \left( \frac{2}{\sqrt{i}} - \frac{2}{\sqrt{i+1}} \right)$$

$$= 2 - \frac{2}{\sqrt{n + 1}}.$$

*Crux Mathematicorum, Vol. 44(8), October 2018*
OC340. Let \( k \) be a fixed positive integer. Alberto and Beralto play the following game: Given an initial number \( N_0 \) and starting with Alberto, they alternately do the following operation: change the number \( n \) for a number \( m \) such that \( m < n \) and \( m \) and \( n \) differ, in their base-2 representation, in exactly \( l \) consecutive digits for some \( l \) such that \( 1 \leq l \leq k \). If someone can’t play, they lose. We say a non-negative integer \( t \) is a winning number when the player who receives the number \( t \) has a winning strategy, that is, they can choose the next numbers in order to guarantee their own victory, regardless of the options of the other player. Otherwise, we call \( t \) a losing number.

Prove that, for every positive integer \( N \), the total of non-negative losing integers less than \( 2^N \) is

\[
2^N - \left\lfloor \frac{\log(\min(N, k))}{\log 2} \right\rfloor.
\]

*Originally Problem 3 from the 2016 Brazil National Olympiad Day 1.*

*We received no solutions for this problem.*