OC336. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $a, b \in \mathbb{N}$, we have

$$(f(a) + b)f(a + f(b)) = (a + f(b))^2.$$ 

Originally Problem 2 from the 2016 Iranian Mathematical Olympiad (3rd Round).

Michel Bataille notified us that OC336 appeared before in our journal as problem 4040 of 41(4) and solved in 42(4). OC336 received two solutions, both of which were correct, complete and different than previously published solutions. We present the solution by Steven Chow.

We assume that both domain and codomain of the function $f$ are the set of all integers greater than or equal to 1. The equation that defines the function $f$ is

$$(f(x) + y)f(x + f(y)) = (x + f(y))^2. \tag{1}$$

Take $x = a$ and $y = a$ in (1) to obtain $$(f(a) + a)f(a + f(a)) = (a + f(a))^2.$$ Since $a + f(a) \geq 1 + 1$, it follows that for all integers $a \geq 1$

$$f(a + f(a)) = a + f(a). \tag{2}$$

Using mathematical induction, we shall prove that the statement

$$P(x) : f(xa) = xf(a) \text{ for all integers } a \geq 1$$

is true for all integers $x \geq 1$. It is obvious that $P(1)$ is true. Assume that for some integer $k \geq 1$, $P(k)$ holds, equivalently $f(ka) = kf(a)$ for all integers $a \geq 1$.

We evaluate (1) at $x = ka$ and $y = a + f(a)$:

$$(f(ka) + a + f(a))f(ka + f(a + f(a))) = (ka + f(a + f(a)))^2. \tag{3}$$

Because of (2) and $f(ka) = kf(a)$, (3) writes as

$$(a + (k + 1)f(a))f((k + 1)a + f(a)) = ((k + 1)a + f(a))^2. \tag{4}$$

We evaluate (1) at $x = (k + 1)a$ and $y = a$:

$$((k + 1)a + a)f((k + 1)a + f(a)) = ((k + 1)a + f(a))^2 \tag{5}$$

The right sides of (4) and (5) are the same, implying that the left sides of (4) and (5) are equal; and taking into account that $f((k + 1)a + f(a)) \geq 1$, it follows that $(a + (k + 1)f(a)) = (f((k + 1)a) + a)$, or equivalently $f((k + 1)a) = f((k + 1)a)$ for any integer $a \geq 1$. Hence $P(k + 1)$ is established; end of induction.

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Therefore for all integers \( x \geq 1 \) and \( a \geq 1 \), \( f(ax) = xf(a) \). Since \( f(x) = f(1)x \) and \( a + f(a) = f(a + f(a)) = f(1)(a + f(a)) \) it follows that \( f(1) = 1 \). Hence for all integers \( x \geq 1 \), \( f(x) = x \). Indeed, this function satisfies the condition (1).

In conclusion, the only solution of (1) is the identity function, \( f(x) = x \) for all integers \( x \geq 1 \).

**OC337.** Find all polynomials \( P(x) \) with integer coefficients such that

\[
P(P(n) + n)
\]

is a prime number for infinitely many integers \( n \).

*Originally Problem 3 from the 2016 Canadian Mathematical Olympiad.*

We received three solutions, all of which were correct and complete. We present the solution by Jose Luis Diaz-Barrero.

Let \( P(x) \) be a polynomial with integer coefficients of degree \( m \). That is,

\[
P(x) = \sum_{k=0}^{m} a_k x^k
\]

with \( a_k \in \mathbb{Z} \) for \( 0 \leq k \leq m \). Next we consider two cases.

First, if \( P(x) \) is a constant polynomial equal to \( p \), then \( P(P(n) + n) = p \) for any integer \( n \). Hence \( P(P(n) + n) \) is a prime number for infinitely many integers \( n \) if and only if \( p \) is a prime number.

Second, assume that \( P(x) \) is not a constant polynomial, hence its degree \( \deg(P) \geq 1 \). We prove that there exists a polynomial \( Q(x) \) such that \( P(P(x) + x) = P(x)Q(x) \) and \( \deg(Q) = (\deg(P))^2 - \deg(P) \).

\[
P(P(x) + x) - P(x) = \sum_{k=0}^{m} a_k (P(x) + x)^k - \sum_{k=0}^{m} a_k x^k
\]

\[
= \sum_{k=1}^{m} a_k ((P(x) + x)^k - x^k)
\]

\[
= \sum_{k=1}^{m} a_k P(x) \sum_{l=0}^{k-1} x^k
\]

\[
= P(x) \sum_{k=1}^{m} \sum_{l=0}^{k-1} a_k x^k = P(x)Q(x),
\]

with \( Q(x) = \sum_{k=1}^{m} \sum_{l=0}^{k-1} a_k x^k \). In addition

\[
\deg(Q) = \deg(P(P(x) + x)) - \deg(P) = (\deg(P))^2 - \deg(P).
\]

Since \( P(P(n) + n) = P(n)Q(n) \) is prime for infinitely many integers \( n \), it follows that either \( P(n) = \pm 1 \) for infinitely many integers \( n \) or \( Q(n) = \pm 1 \) for infinitely many integers \( n \).
many integers \( n \). Since \( P(x) \) is a polynomial of degree \( m \geq 1 \), \( P(n) = \pm 1 \) cannot hold for infinitely many \( n \). So, we have for infinitely many integers \( n \), \( Q(n) = \pm 1 \), equivalently \( \deg(Q) = 0 \). Therefore \( (\deg(P))^2 - \deg(P) = 0 \), \( \deg(P) = 1 \).

Let \( P(x) = a_0 + a_1 x \) for some integers \( a_0 \) and \( a_1 \). Since

\[
P(P(x) + x) = a_0 + a_1(a_0 + a_1 x + x) = P(x)(1 + a_1)
\]

it follows that \( Q(x) = 1 + a_1 = \pm 1 \). The case \( 1 + a_1 = 1 \) leads to \( a_1 = 0 \) which is impossible since we assumed \( \deg(P) \geq 1 \). The case \( 1 + a_1 = -1 \) leads to \( a_1 = -2 \), \( P(x) = a_0 - 2x \), and \( P(P(x) + x) = a_0 - 2x \). Lastly \( P(P(n) + n) = a_0 - 2n \) is prime for infinitely many integers \( n \) if and only if \( a_0 \) is odd.

Finally, we conclude that constant polynomial \( P(x) = p \) with \( p \) a prime number or \( P(x) = a_0 - 2x \) with \( a_0 \) an odd integer are the polynomials \( P(x) \) that fulfill the conditions of the statement.

**OC338.** Let \( \Gamma \) be the excircle of triangle \( ABC \) opposite to the vertex \( A \) (namely, the circle tangent to \( BC \) and to the extensions of the sides \( AB \) and \( AC \) from the points \( B \) and \( C \)). Let \( D \) be the center of \( \Gamma \) and \( E, F \), respectively, the points in which \( \Gamma \) touches the extensions of \( AB \) and \( AC \). Let \( J \) be the intersection between the segments \( BD \) and \( EF \). Prove that \( \angle CJB \) is a right angle.

*Originally Problem 3 from the 2016 Italian Mathematical Olympiad.*

We received six solutions, all of which were correct and complete. Four solutions were geometrical, one solution used complex numbers, and one solution used barycentric coordinates. We present 2 solutions.

**Solution 1**, by Mohammed Aassila.

Denote

\[
\angle A = \angle BAC, \quad \angle B = \angle CBA, \quad \text{and} \quad \angle C = \angle ACB.
\]

Since the angles of a triangle add up to \( 180^\circ \), we obtain for triangle \( BDC \):

\[
\angle BDC = 180^\circ - \angle DCB - \angle CBD = 180^\circ - \frac{180^\circ - \angle C}{2} - \frac{180^\circ - \angle B}{2} = \frac{180^\circ - \angle A}{2}.
\]

Since triangle \( AEF \) is isosceles, we obtain :

\[
\angle JFC = \frac{180^\circ - \angle A}{2}.
\]

So the points \( J, C, F \), and \( D \) lie on the same circle, hence

\[
\angle CJD = 180^\circ - \angle DFC = 90^\circ,
\]

and

\[
\angle CJB = 180^\circ - \angle CJD = 90^\circ.
\]
Solution 2, by Oliver Geupel.

Let $K$ be the point of tangency of the line $BC$ to $\Gamma$. Let $LL'$ be the diameter of $\Gamma$ with the property that the points $B$, $L$, and $L'$ are collinear. Let $M$ be the midpoint of $BC$. Put the picture onto a complex plane such that $\Gamma$ is the unit circle, and identify each point with the corresponding complex number.

It is well-known that the intersection of tangents from tangency points $X$ and $Y$ to the unit circle is the point $2XY/(X+Y)$. Hence,

$$B = \frac{2EK}{E+K}, \quad C = \frac{2FK}{F+K}, \quad M = \frac{B+C}{2} = K\frac{2EF+EK+FK}{(E+K)(F+K)}.$$

Then

$$|B-M|^2 = (B-M)(\overline{B-M}) = \frac{K^2(E-F)}{(E+K)(F+K)} \cdot \frac{\left(\frac{1}{E} - \frac{1}{F}\right)}{\left(\frac{1}{E} + \frac{1}{F}\right) \left(\frac{1}{E} + \frac{1}{K}\right)}$$

$$= -\frac{K^2(E-F)^2}{(E+K)^2(F+K)^2}.$$

Since $\angle KDL = \angle LDE$, we have $L^2 =EK$. It is a well-known and easily verifiable fact that the intersection of chords $\{X,Y\}$ and $\{Z,U\}$ of the unit circle is the point

$$(XY(Z+U) - ZU(X+Y))/(XY-ZU).$$

We obtain for the intersection $J$ of chords $\{E,F\}$ and $\{L,-L\}$:

$$J = L^2\frac{E+F}{EF+L^2} = K\frac{E+F}{EF+EK} = \frac{K(E+F)}{F+K^2},$$

and

$$|J-M|^2 = (J-M)(\overline{J-M}) = \frac{KE(E-F)}{(E+K)(F+K)} \cdot \frac{\left(\frac{1}{E} - \frac{1}{F}\right)}{\left(\frac{1}{E} + \frac{1}{K}\right) \left(\frac{1}{F} + \frac{1}{K}\right)}$$

$$= -\frac{K^2(E-F)^2}{(E+K)^2(F+K)^2} = |B-M|^2.$$
Thus, \( J \) lies on the circle with center \( M \) and radius \( BM \). This circle has \( CB \) as diameter, therefore \( \angle CJB = 90^\circ \).


**OC339.** Let \( n \) be any positive integer. Prove that

\[
\sum_{i=1}^{n} \frac{1}{(i^2 + i)^{3/4}} > 2 - \frac{2}{\sqrt{n + 1}}.
\]

Originally Problem 3 from the 2016 Philippines Mathematical Olympiad.

We received 8 solutions, all of which were complete and correct. 7 solutions used AM-GM inequality, of which 4 solutions were completed via mathematical induction. One solution used the monotonicity of a function. We present a solution that was independently proposed by Mohammad Aassila, Steven Chow, and Oliver Geupel.

For all integers \( j \geq 1 \),

\[
\frac{1}{\sqrt{j}} > \frac{1}{\sqrt{j} + 1} > 0,
\]

so from the AM-GM inequality,

\[
\frac{1}{\sqrt{j}} + \frac{1}{\sqrt{j} + 1} > \frac{2}{(j^2 + j)^{3/4}}.
\]

In the above inequality, multiply both sides by \( 1/(j^2 + j)^{3/4} \) to obtain

\[
\frac{1}{(j^2 + j)^{3/4}} \left( \frac{2}{\sqrt{j}} + \frac{2}{\sqrt{j} + 1} \right) > \frac{4}{j^2 + j} = 4 \frac{j}{j} - 4 \frac{j}{j + 1} = \left( \frac{2}{\sqrt{j}} + \frac{2}{\sqrt{j} + 1} \right) \left( \frac{2}{\sqrt{j}} - \frac{2}{\sqrt{j} + 1} \right),
\]

so

\[
\frac{1}{(j^2 + j)^{3/4}} > \frac{2}{\sqrt{j}} - \frac{2}{\sqrt{j + 1}}.
\]

Therefore

\[
\sum_{i=1}^{n} \frac{1}{(i^2 + i)^{3/4}} > \sum_{i=1}^{n} \left( \frac{2}{\sqrt{i}} - \frac{2}{\sqrt{i + 1}} \right) = 2 - \frac{2}{\sqrt{n + 1}}.
\]

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Let $k$ be a fixed positive integer. Alberto and Beralto play the following game: Given an initial number $N_0$ and starting with Alberto, they alternately do the following operation: change the number $n$ for a number $m$ such that $m < n$ and $m$ and $n$ differ, in their base-2 representation, in exactly $l$ consecutive digits for some $l$ such that $1 \leq l \leq k$. If someone can’t play, they lose. We say a non-negative integer $t$ is a winning number when the player who receives the number $t$ has a winning strategy, that is, they can choose the next numbers in order to guarantee their own victory, regardless of the options of the other player. Otherwise, we call $t$ a losing number.

Prove that, for every positive integer $N$, the total of non-negative losing integers less than $2^N$ is

$$2^N - \left\lfloor \frac{\log_{\log 2} \min\{N, k\}}{\log 2} \right\rfloor.$$ 

*Originally Problem 3 from the 2016 Brazil National Olympiad Day 1.*

*We received no solutions for this problem.*