Focus on...
No. 32
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Harmonic Ranges and Pencils

Introduction

Elementary properties of harmonic conjugacy can lead to simple and elegant solutions to some geometry problems. Before considering examples, let us recall the basic definitions. Let $A, B, C, D$ be four distinct points on a line. We say that $C, D$ are harmonic conjugates w.r.t $A, B$ when $C, D$ divide $AB$ in the same ratio, that is, if $\frac{CA}{CB} = -\frac{DA}{DB}$ (here and in what follows, the bar indicates signed distance and w.r.t. means "with respect to"). Clearly, the latter is equivalent to $\frac{AD}{AC} = -\frac{BD}{BC}$, meaning that $A, B$ are harmonic conjugates w.r.t. $C, D$. If either condition is satisfied, we say that $A, B, C, D$ is a harmonic division or a harmonic range. Let $I$ be the midpoint of $AB$. Starting with $\vec{AD} = -k \vec{AC}$ and $\vec{BD} = k \vec{BC}$ for some real number $k$, easy manipulations give $\vec{ID} = k \vec{IA}$ and $\vec{IA} = k \vec{IC}$ and conversely. Thus, the condition $IA^2 = IC \cdot ID$ can also be used to prove the harmonicity of the range of collinear points $A, B, C, D$.

Harmonic pencil

Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four distinct lines which are either parallel or concurrent, and let transversals $m, m'$ meet them in $A, B, C, D$ and in $A', B', C', D'$, respectively. If $\ell_1, \ell_2, \ell_3, \ell_3$ are parallel, then $\frac{CA'}{CB'} = \frac{CA}{CB}$ and $\frac{DB'}{DB} = \frac{DA}{DB}$ so that $A', B', C', D'$ is a harmonic range as soon as $A, B, C, D$ is one (Figure 1).

This conservation of harmonicity remains true when $\ell_1, \ell_2, \ell_3, \ell_4$ are concurrent lines. To prove this, we shall use the following lemma (for a proof, we refer the reader to [1] p. 169).

Let $A, B, C, D$ be four distinct points on a line and $S$ a point not on this line. Let the parallel to $SC$ through $A$ intersect $SD$ at $M$ and $SB$ at $E$. Then $A, B, C, D$ is a harmonic range if and only if $M$ is the midpoint of $AE$ (Figure 2).
Now, let $\ell_1, \ell_2, \ell_3, \ell_4$ be concurrent at $S$ and let $m$ intersect them along a harmonic range $A, B, C, D$. If $m'$ intersects them along $A', B', C', D'$, we draw the parallels to $SC$ through $A$ and through $A'$, which intersect $SD$ and $SB$, respectively at $M$ and $E$ and at $M'$ and $E'$ (Figure 2). Since $A, B, C, D$ is a harmonic range, $M$ is the midpoint of $AE$; since $AE$ is parallel to $A'E'$, $M'$ is the midpoint of $A'E'$ and so $A', B', C', D'$ is a harmonic range as well.

This justifies the following definition: $\ell_1, \ell_2, \ell_3, \ell_4$ is called a harmonic pencil when a transversal $m$ intersects $\ell_1, \ell_2, \ell_3, \ell_4$ along a harmonic range. From the lemma above, an example is given by the lines $\ell, AM, AB, AC$ if $M$ is the midpoint of the side $BC$ of $\triangle ABC$ and $\ell$ is the parallel to $BC$ through $A$.

We are now ready to examine a few situations involving harmonic ranges or pencils and illustrate them with problems.

**An angle and its bisectors**

If $ABC$ is a triangle (with $AB \neq AC$) and the internal and external bisectors of $\angle BAC$ meet $BC$ at $D$ and $D'$, respectively, we know that $D, D'$ divide $BC$ in the ratio $\frac{AB}{AC}$. Thus, $B, C, D, D'$ is a harmonic range and $AB, AC, AD, AD'$ is a harmonic pencil. Note that $AD, AD'$ are perpendicular. Interestingly, a kind of converse holds (easily proved or see [1] p. 170):

Let $\ell_1, \ell_2, \ell_3, \ell_4$ be a harmonic pencil of concurrent lines at $S$. If $\ell_3, \ell_4$ are perpendicular, then they are the axes of symmetry of $\ell_1, \ell_2$.

To illustrate these results, we consider problem 3036 [2005 : 175 ; 2006 : 244], slightly modified:

Let $A, B, C$ be three distinct collinear fixed points. Let $M$ be an arbitrary point not on the line $ABC$. The internal angle bisector of $\angle MAB$ intersects the line $MB$ at a point $X$. The perpendicular at $A$ to the line $AX$ intersects the line $MC$ at a point $Y$.

(a) Prove that the line $XY$ passes through a fixed point $D$.

(b) Let $Z$ be the projection of the point $A$ onto the line $XY$. Prove that $ZA$ is a bisector of $\angle BZC$.

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(a) Let $XY$ intersect $MA$ at $U$ and the line $m$ through $A, B, C$ at $D$ (Figure 3). Since $AY$ and $AX$ are the bisectors of $\angle MAB$, the pencil $AY, AX, AM, AB$ is harmonic. Therefore, the range $Y, X, U, D$ is harmonic and in consequence the pencil $MY, MB, MU, MD$ is harmonic. Finally, considering the latter and the transversal $m$, we see that $C, B, A, D$ is harmonic and conclude that $XY$ always passes through the harmonic conjugate of $A$ w.r.t. $B, C$.

(b) Since $ZA$ and $ZD$ are perpendicular and $C, B, A, D$ is a harmonic range, $ZA, ZD$ are the bisectors of $\angle BZC$.

**About polars with respect to a circle**

Consider a circle $\Gamma$ with centre $O$ and radius $r$ and let $M$ be a point distinct from $O$. The locus $\Pi_M$ of points $P$ such that the dot product $\overrightarrow{OM} \cdot \overrightarrow{OP}$ equals $r^2$ is the polar of $M$ w.r.t. $\Gamma$. If $M$ lies on $\Gamma$, $M$ itself is a point of $\Pi_M$; otherwise, denoting by $A$ and $B$ the points of intersection of $\Gamma$ and the line $OM$, we see that the harmonic conjugate $M'$ of $M$ w.r.t. $A, B$ is a point of $\Pi_M$ (since $O$ is the midpoint of $AB$ and $\overline{OM} \cdot \overline{OM'} = OA^2$). Moreover, $P$ is on $\Pi_M$ if and only if $\overrightarrow{OM} \cdot \overrightarrow{OP} = \overrightarrow{OM} \cdot \overrightarrow{OM'}$, which is equivalent to $\overrightarrow{OM} \cdot \overrightarrow{M'P} = 0$, and therefore $\Pi_M$ is the perpendicular to $OM$ through $M'$ (Figure 4). In the same way, we obtain that if $M$ is on $\Gamma$, then $\Pi_M$ is the tangent to $\Gamma$ at $M$. 

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Figure 3

Figure 4

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In passing, note that $M'$ is the inverse of $M$ in $\Gamma$ and that $M$ is on $\Pi_N$ as soon as $N$ is on $\Pi_M$ (polar reciprocity).

The result that $M, M', A, B$ is a harmonic range can be generalized as follows:

If a line through $M$ intersects $\Pi_M$ at $N$ and the circle $\Gamma$ at $U$ and $V$, then $M, N, U, V$ is a harmonic range.

Let the perpendicular bisector of $UV$ intersect $UV$ at $I$, $\Pi_M$ at $K$ and the perpendicular to $OM$ through $U$ at $L$ (Figure 4). Clearly, $U$ is the orthocenter of $\Delta OML$, hence $OU$ is perpendicular to $ML$. But $M$ is on $\Pi_K$ (since $K$ is on $\Pi_M$) and so $UV$ is the polar of $K$. It follows that $K$ is on the polar of $U$, that is, the tangent to $\Gamma$ at $U$. Being both perpendicular to $OU$, $ML$ and $KU$ are parallel, and so are $KN$ and $LU$. As a result, if $\vec{IU} = k\vec{IN}$, then $\vec{IL} = k\vec{IK}$ and so $\vec{IM} = k\vec{IU}$. The result follows.

As an application, we propose here a problem of the 50th Olympiad of Moldova [2009 : 377]:

The quadrilateral $ABCD$ is inscribed in a circle. The tangents to the circle at $A$ and $C$ intersect at a point $P$ not on $BD$ and such that $PA^2 = PB \cdot PD$. Prove that $BD$ passes through the midpoint of $AC$.

Let $\Gamma$ be the circumcircle of $ABCD$ and let $O$ be its centre. The line $PD$ intersects $\Gamma$ again at $B'$ with $B' \neq B$ (since $P$ is not on $BD$). Since $PB' \cdot PD$ is the power of $P$ w.r.t. $\Gamma$, we have $PB' \cdot PD = PA^2 = PB \cdot PD$, so that $PB' = PB$ and the line $OP$ is the perpendicular bisector of $BB'$ (Figure 5).

It follows that $OP$ is a bisector of the angle $\angle BPD$ and so is the line $m$ perpendicular to $OP$ at $P$. As a result, $PO, m, PD, PB$ is a harmonic pencil and the line $BD$ intersects $PO$ and $m$ at $Q$ and $S$ such that $Q, S, B, D$ is a harmonic range. From the property above, we then deduce that the polar of $Q$ w.r.t. $\Gamma$ passes through $S$, hence is $m$ (since $m$ is perpendicular to $OQ$). By polar reciprocity, $Q$ is on the polar of $P$, which is $AC$, and the conclusion immediately follows.
Constructions with the straightedge alone

Harmonic ranges or pencils can be constructed with the straightedge alone. This interesting feature rests upon the following property:

Let \( \ell_1, \ell_2 \) be two lines intersecting at \( S \) and \( A \) a point not on these lines. Through \( A \) we draw two transversals intersecting \( \ell_1, \ell_2 \) at \( M_1, M_2 \) and \( N_1, N_2 \). If the lines \( M_1 N_2 \) and \( M_2 N_1 \) intersect at \( U \), then \( SA, SU, \ell_1, \ell_2 \) is a harmonic pencil.

The proof is easy: If \( B \) is the harmonic conjugate of \( A \) w.r.t. \( M_1, M_2 \), the line through \( A, N_1, N_2 \) is a transversal of the harmonic pencil \( UA, UB, UM_1, UM_2 \), hence intersects \( UB \) at \( C \) such that \( A, C, N_2, N_1 \) is a harmonic range. The line \( SB \), which also passes through \( C \), must coincide with \( SU \).

Of course, if \( \ell_1, \ell_2 \) are parallel, a similar conclusion holds provided that \( SA \) and \( SU \) are replaced by the parallels to \( \ell_1, \ell_2 \) through \( A \) and \( U \), respectively.

To see this at work, a good example is Problem 2965 [2004 : 367, 370; 2005 : 405]:

Let \( ABCD \) be a parallelogram. Using only an unmarked straightedge, find a point \( M \) on \( AB \) such that \( AM = \frac{1}{5} AB \).

Here are the steps of the construction. First, we obtain the reflection \( B_1 \) of \( B \) in \( A \) by drawing \( \ell \) such that \( DC, DA, DB, \ell \) is a harmonic pencil (Figure 6a). The line \( \ell \) intersects the line \( AB \) at \( B_1 \). Second, we construct \( B'_1 \) such that \( A, B, B_1, B'_1 \) is a harmonic range (Figure 6b). Finally, we repeat the first two steps with \( B'_1 \) instead of \( B \). This yields the desired point \( M \) as the harmonic conjugate w.r.t. \( A, B \) of the reflection \( B_2 \) of \( B'_1 \) in \( A \).

Indeed, we have \( \frac{B'_1A}{B'_1B} = -\frac{B_1A}{B_1B} = -\frac{1}{2} \) and so

\[
\frac{AM}{MB} = \frac{B_2A}{B_2B} = -\frac{B'_1A}{AB_1 + AB} = -\frac{B'_1A}{B_1'B_1B + AB/B_1'B} = \frac{1/2}{1/2 + 3/2} = \frac{1}{4}
\]

The relation \( AM = \frac{1}{5} AB \) follows.

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Exercises

1. Through a point $P$ exterior to a given circle pass a secant and a tangent to the circle. The secant intersects the circle at $A$ and $B$, and the tangent touches the circle at $C$ on the same side of the diameter through $P$ as $A$ and $B$. The projection of $C$ onto the diameter is $Q$. Prove that $QC$ bisects $\angle AQB$. (Set at the competition Baltic Way in 2004.)

2. The standard construction for bisecting a line segment involves the use of two arcs and one straight line. Show that it can, in fact, be done with straight lines and just one arc. (Problem 88.I of the Mathematical Gazette, proposed in November 2004.)

Reference