No. 5

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This month, we will look at problem A6 from the 2017 Canadian Senior Mathematics Contest, hosted by the The Centre for Education in Mathematics and Computing at the University of Waterloo. You can check out the contest, and past contests on the CEMC website at www.cemc.uwaterloo.ca.

**A6.** In the diagram, the triangle has side lengths 6, 8 and 10. Three semi-circles are drawn using the sides of the triangle as diameters. A large circle is drawn so that it just touches each of the three semi-circles. What is the radius of the large circle?

Over the years, in Ontario at least, the amount of geometry in the high school curriculum has decreased dramatically. All that remains are a few elementary angle properties, some explorations of geometric properties from an analytic geometry point of view and a brief mention of congruent and similar triangles on the way to defining the trigonometric ratios. There remains no study of the geometry of the circle or deductive proofs. In this day where there exists many dynamic geometry software packages like *the Geometer’s Sketchpad* and *Geogebra*, where students could discover, better understand and delve deeper into geometric thinking, it seems like a lost opportunity.

I will now get down off my soapbox, and get down to business. We will need a few geometric definitions and theorems before we begin.

First we need to define a *tangent* to a circle as a line that just touches the circle at a single point, called the *point of tangency*. Any other line that passes through the point of tangency will intersect the circle at another point. In the diagram below, line $\ell_1$ is tangent to the circle at $P$, while $\ell_2$ is not, and thus passes through a second point $Q$ on the circle.
A useful property of tangents to circles is given in the following theorem.

**Theorem 1:** Any line $\ell$ tangent to a circle is perpendicular to the radius at the point of tangency.

In the diagram above, radius $OP$ is perpendicular to the tangent at $P$, $\ell_1$. We can extend this idea and talk about two circles being tangent if they too just touch at one point. They can be internally tangent if one circle is inside the other or externally tangent otherwise. The fact that lines tangent to circles are perpendicular to the radius at the point of tangency leads to the fact that tangent circles will share a tangent line at their point of tangency. This leads to the following theorem that we will need to solve our problem.

**Theorem 2:** If two circles with centres $O_1$ and $O_2$ are tangent to each other at a point $P$, then $O_1$, $O_2$ and $P$ are collinear.

As an example, in the diagram below circles with centres $O_1$ and $O_2$ are internally tangent at $P$ while circles with centres $O_1$ and $O_3$ are externally tangent at $P$. Note that all the centres and the common point of tangency lie on $\ell_1$ while $\ell_2$ is a common tangent for the three circles.

Now on to the solution of our problem. First, since the triangle has sides 6, 8 and 10 units, the Pythagorean theorem shows that this is a right triangle.

In the diagram, $O_r$ and $P_r$ represent the centre of the semicircle with radius $r$ and its point of tangency with the outer circle, respectively, while $O$ represents the centre of the outer circle. For $r = 3, 4, 5$: $P_r$, $O_r$, and $O$ are collinear, by theorem 2. Thus we can draw in radii for the outer circle to the points of tangency with the semicircles.
Let $r$ represent the desired radius and $s$ and $t$ represent the distance from the centre of the large circle to the legs of the triangle, as indicated in the diagram. From the two shaded right triangles we get

$$\begin{align*}
(3 - s)^2 + t^2 &= (r - 3)^2 \quad (1) \\
S^2 + (4 - t)^2 &= (r - 4)^2. \quad (2)
\end{align*}$$

We need the following property of triangles: when the midpoints of two sides of a triangle are joined, the resulting segment is parallel to, and half the length, of the third side of the triangle. Thus, if we join $O_3$ to $O_5$ and $O_4$ to $O_5$ the resulting segments are perpendicular. We can then create another right angled triangle (shaded in the diagram below) which yields

$$\begin{align*}
(3 - s)^2 + (4 - t)^2 &= (r - 5)^2. \quad (3)
\end{align*}$$
Subtracting (3) from (1) yields
\[ t^2 - (16 - 8t + t^2) = (r^2 - 6t + 9) - (r^2 - 10t + 25) \]
\[ 8t - 16 = 4r - 16 \]
\[ t = \frac{r}{2}. \tag{4} \]

Similarly, subtracting (3) from (2) will eventually give us
\[ s = \frac{r}{3}. \tag{5} \]

Substituting (4) and (5) into (1) to yields
\[ \left(3 - \frac{r}{3}\right) + \left(\frac{r}{2}\right) = (r - 3)^2 \]
\[ 9 - 2r + \frac{r^2}{9} + \frac{r^2}{4} = r^2 - 6r + 9 \]
\[ 4r - \frac{23}{36}r^2 = 0 \]
\[ \frac{23}{36}r \left(\frac{144}{23} - r\right) = 0 \]

which yields \( r = 0 \) and \( r = \frac{144}{23} \). Clearly \( r > 0 \), hence \( r = 0 \) is impossible, thus the desired radius is \( \frac{144}{23} \).

The two theorems stated in this article come from book III of Euclid’s Elements. Theorem 1 is Proposition 18, the converse to Theorem 1 is Proposition 19 and Theorem 2 is Proposition 11. Professor David E. Joyce of the Department of Mathematics and Computer Science at Clark University in Worcester, Massachusetts has created an online, interactive version of Euclid’s Elements. You can access his work at https://mathcs.clarku.edu/djoyce/java/elements/elements.html. My thanks to editor Chris Fisher for pointing out the references and making suggestions that improved this column from its initial draft.

Circles are such simple figures, but they have many interesting properties. We will explore other properties of circles in a future column.