CONTEST CORNER
SOLUTIONS


Problems in this Contest Corner came from Purple Comet! Math Meet.

Collaborative problem solving is more important than ever. The Purple Comet! Math Meet allows middle and high school students from across the globe to form teams and enjoy solving challenging problems with their peers during this online mathematics competition conducted annually since 2003.

The contest is free and last year was accessed by nearly 3,200 teams (over 12,000 students) from 59 countries and translated into 22 different languages. There is a ten-day window during which teams may compete choosing a start time most convenient for them. The problems range in difficulty from fairly easy to extremely challenging. To see past contests and information about how to register your own team for next year’s competition, visit https://purplecomet.org/.

This contest is free due to the generosity of its sponsor, AwesomeMath, which is devoted to providing enriching experiences in mathematics for intellectually curious learners. For more information, visit awesomemath.org.

CC276. The figure below shows a 90 \times 90 square with each side divided into three equal segments. Some of the endpoints of these segments are connected by straight lines. Find the area of the shaded region.

\begin{figure}
\centering
\includegraphics[width=0.2\textwidth]{shaded_region}
\end{figure}

Originally problem 16 from Purple Comet! Math Meet, April 2013.

We received five submissions, of which four were correct. Konstantine Zelator and Ieko Dimitrić provided generalizations. We present the solution given by Andrea Fanchini.
We use cartesian coordinates. From the triangle $EFG$ we have $30 : 60 = x : 60 - x$, so $x = 20$ and hence the points $A$ and $D$ have coordinates $A(70, 20)$ and $D(20, 70)$.

Similarly from the triangle $HOG$ we have $60 : 90 = y : 90 - y$, so $y = 36$ and the points $B$ and $C$ have coordinates $B(54, 36)$ and $C(36, 54)$.

Then the triangles $AFG$ and $DIL$ have area $A_1 = \frac{60 \cdot 20}{2} = 600$ and the triangles $BFO$ and $CIO$ have area $A_2 = \frac{30 \cdot 36}{2} = 540$. Therefore, the entire white region has area

$$A_{\text{white}} = 4(A_1 + A_2) = 4560$$

and finally the area of the shaded region is

$$A_{\text{shaded}} = A_{\text{square}} - A_{\text{white}} = 8100 - 4560 = 3540.$$

**CC277.** Let $A, B, C, D, E, F, G$ and $H$ be the eight vertices of a $30 \times 30 \times 30$ cube as shown.

The two figures $ACFH$ and $BDEG$ are congruent regular tetrahedra. Find the volume of the intersection of these two tetrahedra.

*Originally problem 28 from Purple Comet! Math Meet, April 2013.*

We received one correct submission, by Ivko Dimitrić, and we present it here.

It is well-known that each convex polyhedron is the intersection of half-spaces of the planes determined by its faces. Conversely, a bounded intersection of a finite number of half-spaces is a convex polyhedron.

Let $V$ and $W$ be the centers of the top and the bottom faces of the cube, respectively, and let $P, Q, R, S$ be the centers of the lateral faces of the cube in cyclic order, starting with $P$ as the center of the face $AEHD,$ followed by $Q,$ the center of the face $AEFB.$

Since the edges of the two tetrahedra can be paired into six pairs (one edge from each tetrahedron in a pair) so that the two edges in each pair are the face diagonals of one of the six faces of the cube, the points $P, Q, R, S, V$ and $W$ belong to both tetrahedra, so the convex hull of these six points, which is a regular octahedron with its vertices being these six points, belongs to the (convex) intersection II of the two tetrahedra.
We argue that the intersection \( \Pi \) is, in fact, that octahedron.

If \( O \) is the center of the cube, the segments joining pairs of opposite vertices of the octahedron meet at \( O \) perpendicularly and \( O \) bisects each of them. Thus the cube and the octahedron have the same center.

Each face of the octahedron is the medial triangle of one of the combined eight faces of the two tetrahedra, such as the face \( QRV \) being the medial triangle of \( ACF \) and similarly for others. Hence, the plane supporting \( QRV \) is the same as the plane \( ACF \), and the octahedron is located on that side of the plane \( (QRV) = (ACF) \) that contains \( O \). A similar situation occurs for other planes determined by the faces of the octahedron. Thus, the intersection \( \Pi \) of the two tetrahedra, i.e. the intersection of eight half-spaces containing \( O \), determined by the faces of these tetrahedra, is identical to the intersection of eight half-spaces containing \( O \), determined by the planes supporting the faces of the octahedron, i.e. the solid \( \Pi \) is the octahedron.

The edge \( QR \) of this octahedron is the common midline of triangles \( BEG \) and \( AFC \) and its length is half the length of \( EG \), i.e. equal to \( \frac{a \sqrt{2}}{2} \), where \( a = 30 \) is the edge length of the cube. The same is true of the lengths of other edges, each face of the octahedron being an equilateral triangle.

The octahedron is composed of two congruent pyramids sharing the same square base \( PQRS \), each having height of \( \frac{a}{2} = 15 \), the half-length of the edge of the cube. The area of the common base of the two pyramids is

\[
QR^2 = \frac{a}{\sqrt{2}} \cdot \frac{a}{\sqrt{2}} = \frac{a^2}{2},
\]

so the volume of the octahedron is

\[
2 \cdot \frac{1}{3} \cdot \frac{a^2}{2} \cdot \frac{1}{2} a = \frac{1}{6} a^3 = \frac{1}{6} \cdot 30^3 = 4500.
\]

**CC278.** For positive integers \( m \) and \( n \), the decimal representation for the fraction \( \frac{m}{n} \) begins with 0.711 and is followed by other digits. Find the least possible value for \( n \).

*Originally problem 17 from Purple Comet! Math Meet, April 2013.*

We received two correct solutions and one incomplete solution. We provide the solution by Titu Zvonaru.

The answer is \( n = 45 \).

Since \( 32/45 = 0.7\overline{1} \), the least possible value for \( n \) cannot exceed 45. The number \( m/n \) begins with 0.711 if and only if \( 711n \leq 1000 < 712n \). It is straightforward to check that no multiple of 1000 lies in the interval \([711n, 712n]\) for \( 1 \leq n \leq 44 \).

**CC279.** There is a pile of eggs. Joan counted the eggs, but her count was off by 1 in the 1’s place. Tom counted the eggs, but his count was off by 1 in the
10’s place. Raoul counted the eggs, but his count was off by 1 in the 100’s place. Sasha, Jose, Peter, and Morris all counted the eggs and got the correct count. When these seven people added their counts together, the sum was 3162. How many eggs were in the pile?

*Originally problem 19 from Purple Comet! Math Meet, April 2013.*

We received 5 submissions, of which 2 were correct and complete and 3 were incomplete. We present the solution by Ivko Dimitrić.

We do not consider changing digit 9 to 0 or 0 to 9 as being off by 1, since in this case one would be off by 9. So for any digit $d$, 1 through 8, being off by 1 means that that digit is given as $d \pm 1$, whereas when $d = 9$ being off by 1 means the digit is changed to $d - 1 = 8$ and when $d = 0$ being off by 1 means the digit is changed to $d + 1 = 1$. Since the average of the seven counts is about 451.7 and the error is at most $\pm 111$, the number of eggs is some 3-digit number

$$N = abc = 100a + 10b + c,$$

with $a$ being the digit of hundreds, $b$ the digit of tens and $c$ the digit of ones. Then Joan’s, Tom’s and Raoul’s counts are respectively

$$J = 100a + 10b + c \pm 1, \quad T = 100a + 10(b \pm 1) + c, \quad R = 100(a \pm 1) + 10b + c,$$

with the proviso that if one of the digits is 0 or 9 the error can go only one way. Then adding these three together with the four correct counts yields

$$4N + 3(100a + 10b + c) \pm 100 \pm 10 \pm 1 = 3162.$$

Therefore,

$$7N \pm 100 \pm 10 \pm 1 = 3162 = 7 \cdot 452 - 2.$$

It follows that

$$2 \pm 100 \pm 10 \pm 1 = 7(452 - N)$$

is divisible by 7. Since $10 \equiv 3 (\text{mod} \, 7)$ and $100 \equiv 2 (\text{mod} \, 7)$, that means that $2 \pm 2 \pm 3 \pm 1$ must be divisible by 7 with an appropriate choice of signs.

Considering all possible choices for $+$ and $-$ signs we see that this is possible only for $2 + 2 - 3 - 1 = 0$ combination. With this choice of signs, Joan’s count was off by 1 by under-counting by 1 the units, Tom’s count was off by 1 in tens by under-counting by 10. and Raoul’s count was off by 1 in hundreds by over-counting by 100. Then

$$7N \pm 100 - 10 - 1 = 3162,$$

from where $N = 439$ is the true number of eggs in the pile, whereas Joan’s, Tom’s and Raoul’s counts were respectively

$$J = 438, \quad T = 429 \quad \text{and} \quad R = 539.$$

**CC280.** You can tile a $2 \times 5$ grid of squares using any combination of three types of tiles: single unit squares, two side by side unit squares, and three unit
squares in the shape of an L. The diagram below shows the grid, the available tile shapes, and one way to tile the grid.

In how many ways can the grid be tiled?

*Originally problem 29 from Purple Comet! Math Meet, April 2013.*

We received two solutions, one of which was correct. We present the solution by Missouri State University Problem Solving Group, modified by the editor.

More generally, we find a closed form for the number of tilings of a $2 \times n$ grid. To do so we develop a recurrence relation. Let $A_n$ denote the number of ways of tiling a $2 \times n$ grid; let $B_n$ denote the number of ways of tiling a $2 \times n$ grid with the lower-left square removed.

Consider the upper-left square of a $2 \times n$ grid. It can be covered with a $1 \times 1$ tile, resulting in $B_n$ ways of tiling the remainder of the grid. It can be covered with a horizontal $1 \times 2$ tile, resulting in two possibilities for covering the lower-left square. Using a $1 \times 1$ tile results in $B_{n-1}$ ways of tiling the remainder of the grid. Using a horizontal $1 \times 2$ tile results in $A_{n-2}$ ways of tiling the remainder of the grid. It can be covered with a vertical $1 \times 2$ tile, resulting in $A_{n-1}$ ways of tiling the remainder of the grid. Finally, there are three ways of covering the square with an L-tromino. One forces the lower-left square to be covered with a $1 \times 1$ tile, resulting in $A_{n-2}$ ways of tiling the remainder of the grid. The other two give $B_{n-1}$ ways of tiling the remainder of the grid. Each of these cases is illustrated below.

In summary, we have the following recurrence relation for $A_n$:

$$A_n = A_{n-1} + 2A_{n-2} + B_n + 3B_{n-1}.$$
Therefore, we have the following recurrence relation for \( B_n \)
\[
B_n = A_{n-1} + A_{n-2} + B_{n-1}.
\]
Substituting this expression into our recurrence relation for \( A_n \) yields
\[
A_n = 2A_{n-1} + 3A_{n-2} + 4B_{n-1}.
\]
One easily verifies that \( A_1 = 2, A_2 = 11, \text{ and } B_2 = 4. \)
Applying our double recurrence yields
\[
A_3 = 44, B_3 = 17; \quad A_4 = 189, B_4 = 72; \quad \text{and} \quad A_5 = 798, B_5 = 305.
\]
Therefore, the answer to the original question is 798 tilings.

We now derive a closed form for \( A_n \). Our recurrence relation can be rewritten in matrix form as
\[
\begin{bmatrix}
A_n \\
A_{n-1} \\
B_n
\end{bmatrix}
= \begin{bmatrix}
2 & 3 & 4 \\
1 & 0 & 0 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
A_{n-1} \\
A_{n-2} \\
B_{n-1}
\end{bmatrix}.
\]
It is well-known that \( A_n \) can be expressed as a linear combination of powers of the eigenvalues of the multiplying matrix. In this case, the characteristic polynomial is
\[
\lambda^3 - 3\lambda^2 - 5\lambda - 1
\]
with roots \( \lambda = -1, 2 \pm \sqrt{5} \). Therefore
\[
A_n = c_1(-1)^n + c_2\left(2 + \sqrt{5}\right)^n + c_3\left(2 - \sqrt{5}\right)^n.
\]
Using the known values of \( A_1, A_2, \text{ and } A_3 \) and solving, we find
\[
c_1 = \frac{1}{2}, c_2 = \frac{5 + 3\sqrt{5}}{20}, c_3 = \frac{5 - 3\sqrt{5}}{20}.
\]
In general we have that
\[
A_n = \frac{10(-1)^n + (5 + 3\sqrt{5})\left(2 + \sqrt{5}\right)^n + (5 - 3\sqrt{5})\left(2 - \sqrt{5}\right)^n}{20}.
\]

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