Steep and Shallow Functions
Aditya Guha Roy

1 Introduction

We first introduce the following definition for real numbers $a, b, c$ and $d$:

- $(a, b)$ and $(c, d)$ are similarly sorted if $(a - b)(c - d) \geq 0$,
- $(a, b)$ and $(c, d)$ are oppositely sorted if $(a - b)(c - d) \leq 0$.

**Definition 1** A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is steep if the pairs $(f(x) \cdot x, f(y) \cdot y)$ and $(\frac{f(x)}{x}, \frac{f(y)}{y})$ are always similarly sorted.

**Definition 2** A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is shallow if the pairs $(f(x) \cdot x, f(y) \cdot y)$ and $(\frac{f(x)}{x}, \frac{f(y)}{y})$ are always oppositely sorted.

Geometrically, if the graph of $f$ passes through the intersection of a radial line $y = kx$ and the right hyperbola $xy = c$ (which divide the positive quadrant into four sections), then if $f$ is shallow, it always passes from the left section to the right section, whereas if $f$ is steep it always passes between the upper and lower sections.

For differentiable functions we have the following equivalent criteria:

- A differentiable function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is steep if and only if $\forall x \in \mathbb{R}^+$, $|f'(x)| > \frac{f(x)}{x}$.
- A differentiable function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is shallow if and only if $\forall x \in \mathbb{R}^+$, $|f'(x)| < \frac{f(x)}{x}$.

The proofs are left as an exercise for the readers.

2 Inequalities for steep and shallow functions

**Lemma 1** Let $a, b, c, d$ be real numbers such that the pairs $(a, b)$ and $(c, d)$ are similarly sorted. Then we must have $ac + bd \geq ad + bc$.

Proof: Since the pairs $(a, b)$ and $(c, d)$ are similarly sorted, it follows that

$(a - b)(c - d) \geq 0, \iff (ac + bd) - (ad + bc) \geq 0.$

Now using this lemma we prove some interesting results about steep and shallow functions.
Proposition 1 If \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) is shallow then, for all real numbers \( x, y \) we have
\[
2 \leq \frac{g(x)}{g(y)} + \frac{g(y)}{g(x)} \leq \frac{x}{y} + \frac{y}{x}.
\]

Proof: As \( xy > 0 \), the pairs \( (g(x) \cdot x^2 y, g(y) \cdot y^2 x) \) and \( (g(x) \cdot x, g(y) \cdot y) \) must be similarly sorted. Furthermore since \( g \) is shallow the pairs
\[
\left( g(x) \cdot x^2 y, g(y) \cdot y^2 x \right) \text{ and } \left( \frac{g(x)}{x}, \frac{g(y)}{y} \right).
\]
Thus by (1), we have
\[
g(x) \cdot x^2 y \cdot \frac{g(x)}{x} + g(y) \cdot y^2 x \cdot \frac{g(y)}{y} \leq g(x) \cdot x^2 y \cdot \frac{g(y)}{y} + g(y) \cdot y^2 x \cdot \frac{g(x)}{x},
\]
and thus we get
\[
xy \cdot ((g(x))^2 + (g(y))^2) \leq g(x)g(y) \cdot (x^2 + y^2).
\]
Now dividing both sides by the positive real number \( g(x)g(y)xy \) yields the required inequality. \( \square \)

Proposition 2 If \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is steep then for all real numbers \( x, y \)
\[
\frac{x}{y} + \frac{y}{x} \leq \frac{f(x)}{f(y)} + \frac{f(y)}{f(x)}
\]
(The proof is similar to that of Proposition 1.)

3 Problems

Problem 1 For all positive reals \( a, b \) and all non-negative real numbers \( c \) prove:
\[
\frac{a + c}{b + c} + \frac{b + c}{a + c} \leq \frac{a}{b} + \frac{b}{a}.
\]

Problem 2 For all positive \( a, b \) and all nonnegative \( c \) prove that
\[
\frac{a^{1+c}}{b^{1+c}} + \frac{b^{1+c}}{a^{1+c}} \geq \frac{a}{b} + \frac{b}{a}.
\]

Problem 3 For all acute angles \( A, B \) prove that
\[
\frac{\sin A}{\sin B} + \frac{\sin B}{\sin A} \leq \frac{A}{B} + \frac{B}{A} \leq \frac{\tan A}{\tan B} + \frac{\tan B}{\tan A}.
\]

Problem 4 For all \( x > 1, y > 1 \) show that
\[
\frac{x^y}{y^x} + \frac{y^x}{x^y} \geq \frac{x}{y} + \frac{y}{x}.
\]

Crux Mathematicorum, Vol. 44(6), June 2018
Hints

**Hint for Problem 1** Note that the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(x) = x + c$ is shallow over the set of all positive real numbers.

**Hint for Problem 2** Note that the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(x) = x^{1+c}$ is steep over the set of all positive real numbers.

**Hint for Problem 3** Consider the steepness and shallowness of the functions $\sin(x)$ and $\tan(x)$. (See [1] for the full solution.)

**Hint for Problem 4** Consider the function $f(x) = x^x$.

References