OLYMPIAD SOLUTIONS


OC326. Let \( n \) be a positive integer. Mary writes the \( n^3 \) triples of not necessarily distinct integers, each between 1 and \( n \) inclusive on a board. Afterwards, she finds the greatest number (possibly more than one) in each triple, and erases the rest. For example, in the triple (1, 3, 4) she erases the numbers 1 and 3, and in the triple (1, 2, 2) she erases only the number 1. Show after finishing this process, the amount of remaining numbers on the board cannot be a perfect square.


We received 2 solutions. We present the solution by Steven Chow.

Each triple has either 3 distinct numbers, or 2 equal numbers and 1 other number (by the obvious bijection, \( \frac{1}{2} \) of these triples have exactly 1 greatest number and \( \frac{1}{2} \) of them have exactly 2 greatest numbers), or 3 equal numbers.

By basic counting, therefore the amount of remaining numbers on the board is

\[
1 \cdot n (n - 1) (n - 2) + 2 \cdot \frac{3}{2} \cdot n (n - 1) + 1 \cdot \frac{3}{2} \cdot n (n - 1) + 3 \cdot n
= \frac{1}{2} n (n + 1) (2n + 1)
= \left( \frac{n + 1}{2} \right) (2n + 1).
\]

If that number is a perfect square, since \( \binom{n+1}{2} \) and \( 2n + 1 \) are coprime, then \( \binom{n+1}{2} \) and \( 2n + 1 \) are both perfect squares, so for some integer \( a \geq 1 \), we have

\[
2n + 1 = (2a + 1)^2 \quad \implies \quad n = 2a^2 + 2a,
\]

so

\[
\binom{n + 1}{2} = a (a + 1) (2a^2 + 2a + 1).
\]

Since \( a, a + 1 \) and \( 2a^2 + 2a + 1 \) are pairwise coprime, then \( a \) and \( a + 1 \) are perfect squares, which is a contradiction since \( a \geq 1 \). Therefore the amount of remaining numbers on the board cannot be a perfect square.

OC327. Quadrilateral \( APBQ \) is inscribed in circle \( \omega \) with \( \angle P = \angle Q = 90^\circ \) and \( AP = AQ < BP \). Let \( X \) be a variable point on segment \( PQ \). Line \( AX \) meets \( \omega \) again at \( S \) (other than \( A \)). Point \( T \) lies on arc \( AQB \) of \( \omega \) such that \( XT \) is perpendicular to \( AX \). Let \( M \) denote the midpoint of chord \( ST \). As \( X \) varies on segment \( PQ \), show that \( M \) moves along a circle.

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Originally 2015 USAMO Day 1 Problem 2.

We received 1 solution. We present the solution by Steven Chow.

The following proof also proves that the problem is true without the condition that \( AQ < BP \) and without the condition that \( T \) is on the arc \( AQB \) (as long as \( T \in \omega \)).

Let the unit circle be \( \omega \) such that 1 is on \( A \). Let the corresponding lowercase letter of the name of a point be the complex number of the point.

Since \( \angle APB = \frac{1}{4} \pi \), \( AB \) is a diameter of \( \omega \). Since \( AP = AQ \), \( q = p = \frac{1}{p} \). Since \( X \in PQ \), therefore \( x + \overline{x} = p + \overline{p} \).

Since \( s \neq 1 \) and \( \overline{x} = \frac{1}{s} \) and \( A, S, \) and \( X \) are collinear,

\[
\frac{s - 1}{x - 1} = \frac{1}{s} - 1 = \frac{-s + 1}{s(x - 1)} \implies s = \frac{-x + 1}{\overline{x} - 1}.
\]

Since \( t = \frac{1}{x} \) and \( \angle AXT = \frac{1}{4} \pi \),

\[
0 = \frac{t - x}{x - 1} + \frac{1 - \overline{x}}{\overline{x} - 1} \implies \frac{t}{-x + 1} + \frac{1}{t(-\overline{x} + 1)} = \frac{x + \overline{x} - 2x\overline{x}}{(x - 1)(\overline{x} - 1)}.
\]

Let \( C \) be the point at \( \frac{1}{2} \). It shall be proved that \( M \) is on the circle with centre \( C \) and radius \( CP \). It suffices to prove that \( CM = CP \), which is equivalent to

\[
\iff 0 = CM^2 - CP^2 = (m - c)(\overline{m} - \overline{c}) - (p - c)(\overline{p} - \overline{c})
\]

\[
= \left( \frac{s + t}{2} - 1 \right) \left( \frac{1}{s} + \frac{1}{t} - 1 \right) - \left( p - \frac{1}{2} \right) \left( \frac{1}{p} - \frac{1}{2} \right)
\]

\[
\iff 0 = (s + t - 1) \left( \frac{1}{s} + \frac{1}{t} - 1 \right) - (2p - 1) \left( \frac{2}{p} - 1 \right)
\]

\[
= t \left( \frac{\overline{x} - 1}{x - 1} - 1 \right) + \frac{1}{t} \left( \frac{-x + 1}{\overline{x} - 1} - 1 \right) - \frac{-x + 1}{\overline{x} - 1} - \frac{\overline{x} - 1}{x - 1} + 2p + \frac{2}{p} - 2
\]

\[
= (x + \overline{x} - 2) \left( \frac{t}{-x + 1} + \frac{1}{t(-\overline{x} + 1)} \right) + \frac{(x - 1)^2 + (\overline{x} - 1)^2}{(x - 1)(\overline{x} - 1)} + 2p + \frac{2}{p} - 2
\]

\[
= (x + \overline{x} - 2) \cdot \frac{x + \overline{x} - 2x\overline{x}}{(x - 1)(\overline{x} - 1)} + \frac{(x - 1)^2 + (\overline{x} - 1)^2}{(x - 1)(\overline{x} - 1)} + 2p + \frac{2}{p} - 2
\]

\[
= \frac{-2x^2\overline{x} - 2x\overline{x}^2 + 2x^2 + 6x\overline{x} + 2\overline{x}^2 - 4x - 4\overline{x} + 2}{(x - 1)(\overline{x} - 1)} + 2p + \frac{2}{p} - 2
\]

\[
= 2(-x - \overline{x} + 1) + 2p + \frac{2}{p} - 2
\]

\[
= 0.
\]

Therefore as \( X \) varies on \( PQ \), \( M \) moves along a circle.

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OC328. We call a divisor $d$ of a positive integer $n$ special if $d + 1$ is also a divisor of $n$. Prove: at most half the positive divisors of a positive integer can be special. Determine all positive integers for which exactly half the positive divisors are special.

*Originally from the 2015 South Africa National Olympiad.*

We prove that no positive divisor $d$ of $n$ that is greater or equal to $\sqrt{n}$ can be special: if $d$ is special, then $d + 1$ is also a divisor, so $\frac{n}{d}$ and $\frac{n}{d} + 1$ are both integers, which means that their difference is at least 1. Thus

$$\frac{n}{d} \geq \frac{n}{d + 1} + 1,$$

which is equivalent to $n \geq d(d + 1)$. But since $d(d + 1) > d^2 \geq n$, this is a contradiction. Thus only divisors less than $\sqrt{n}$ can be special.

Since divisors come in pairs (a and $n/a$) such that one of them is less than $\sqrt{n}$ and one greater than $\sqrt{n}$ (when $n$ is a square, $\sqrt{n}$ is paired with itself), this means that at most half the divisors can be special.

If precisely half the divisors are special, then $n$ cannot be a square and every divisor less than $\sqrt{n}$ has to be special. Thus 1 has to be a special divisor, meaning that 2 is a divisor (and thus also special), so 3 is a divisor and so on, up to the greatest integer $k$ that is less than $\sqrt{n}$.

Finally, $k$ is special, so $k + 1$ has to be a divisor as well. Since $k$ is the greatest divisor less than $\sqrt{n}$ and $k + 1$ the least divisor greater than $\sqrt{n}$, their product must be $n$, so

$$n = k(k + 1) = k^2 + k.$$

Moreover, $k - 1$ is also a divisor of $n = k^2 + k$ (unless $k = 1$), so it also divides

$$n - (k - 1)(k + 2) = k^2 + k - (k^2 + k - 2) = 2.$$

This leaves us with $k = 1$, $k = 2$ and $k = 3$ as the only possibilities, giving us $n = 2$, $n = 6$ or $n = 12$.

OC329. Let $n \geq 5$ be a positive integer and let $A$ and $B$ be sets of integers satisfying the following conditions:

1. $|A| = n$, $|B| = m$ and $A$ is a subset of $B$

2. For any distinct $x, y \in B$, $x + y \in B$ iff $x, y \in A$

Determine the minimum value of $m$.

*Originally 2015 China National Olympiad Day 1 Problem 3.*

*No solutions were submitted.*
OC330. Solve the following equation in nonnegative integers:

\[(2^{2015} + 1)^x + 2^{2015} = 2^y + 1\]


We received 2 solutions. We present the solution by José Luis Díaz-Barrero.

If \(x = 0\) then we get \(y = 2015\) and if \(x = 1\) then we obtain \(y = 2016\). Suppose that \(x > 1\). Since

\[2^{2015} + 1 \equiv (-1)^{2015} + 1 \equiv 0 \pmod{3},\]

we have

\[2^y + 1 = (2^{2015} + 1)^x + 2^{2015} \equiv 2^{2015} \equiv 5 \pmod{9},\]

from which \(2^y \equiv 4 \pmod{9}\). Since \(2^6 \equiv 1 \pmod{9}\), then \(2^y \equiv 4 \pmod{9}\) gives \(y = 6k + 2\) for some positive integer \(k\). Working modulo 13, we have

\[2^y + 1 = (2^6)^k \cdot 2^2 + 1 \equiv \pm 1 \equiv \pm 4 + 1 \pmod{13},\]

that is

\[2^y + 1 \equiv 5 \pmod{13} \quad \text{or} \quad 2^y + 1 \equiv -3 \pmod{13}.\]

On the other hand, since \(2^{2015} \equiv 7 \pmod{13}\) then

\[8^x + 7 \equiv 5 \pmod{13} \quad \text{or} \quad 8^x + 7 \equiv -3 \pmod{13}.\]

Both cases are impossible because the remainders of \(8^x\) modulo 13 are 1, 5, 8 and 12 respectively. Finally, we conclude that the only solutions to the given equation are \((0, 2015)\) and \((1, 2016)\).

Editor’s comments. Konstantine Zelator generalized the problem and proved that the equation

\[(2^n + 1)^x + 2^n = 2^y + 1,\]

where \(n\) is a positive integer, \(n \equiv 5 \pmod{6}\), has two nonnegative integer solutions \((0, n)\) and \((1, n + 1)\). So, the proposed problem is the case \(n = 2015\).