The 2018 Canadian Mathematical Olympiad Exam
Official Problem Set

1. Consider an arrangement of tokens in the plane, not necessarily at distinct points. We are allowed to apply a sequence of moves of the following kind: Select a pair of tokens at points $A$ and $B$ and move both of them to the midpoint of $A$ and $B$.

We say that an arrangement of $n$ tokens is collapsible if it is possible to end up with all $n$ tokens at the same point after a finite number of moves. Prove that every arrangement of $n$ tokens is collapsible if and only if $n$ is a power of 2.

2. Let five points on a circle be labelled $A, B, C, D, E$ in clockwise order. Assume $AE = DE$ and let $P$ be the intersection of $AC$ and $BD$. Let $Q$ be the point on the line through $A$ and $B$ such that $AQ = DP$. Similarly, let $R$ be the point on the line through $C$ and $D$ such that $DR = AP$. Prove that $PE$ is perpendicular to $QR$.

3. Two positive integers $a$ and $b$ are prime-related if $a = pb$ or $b = pa$ for some prime $p$. Find all positive integers $n$, such that $n$ has at least three divisors, and all the divisors can be arranged without repetition in a circle so that any two adjacent divisors are prime-related.

Note that 1 and $n$ are included as divisors.

4. Find all polynomials $p(x)$ with real coefficients that have the following property: There exists a polynomial $q(x)$ with real coefficients such that

$$p(1) + p(2) + p(3) + \cdots + p(n) = p(n)q(n)$$

for all positive integers $n$.

5. Let $k$ be a given even positive integer. Sarah first picks a positive integer $N$ greater than 1 and proceeds to alter it as follows: every minute, she chooses a prime divisor $p$ of the current value of $N$, and multiplies the current $N$ by $p^k - p^{k-1}$ to produce the next value of $N$. Prove that there are infinitely many even positive integers $k$ such that, no matter what choices Sarah makes, her number $N$ will at some point be divisible by 2018.