CONTEST CORNER
SOLUTIONS


CC271. Warren’s lampshade has an interesting design. Within a regular hexagon (six sides) are two intersecting equilateral triangles, and within them is a circle which just touches the sides of the triangles. (See the diagram.) The points of the triangles are at the midpoints of the sides of the hexagons.

If each side of the hexagon is 20 cm long, find:

a) the area of the hexagon;

b) the area of each large equilateral triangle;

c) the area of the circle.

Originally Question 5 from the 2011 University of Otago Junior Mathematics Competition.

We received two submissions to this problem, both of which were correct. We present the solution by David Manes, modified by the editor.

a) The formula for the area $A_H$ of a regular hexagon is $A_H = \frac{3\sqrt{3}}{2} s^2$, where $s$ is the side length. Since $s = 20$ cm, we have

$$A_H = \frac{3\sqrt{3}}{2} (20)^2 = 600\sqrt{3} \text{ cm}^2.$$

b) Note that the two large equilateral triangles are congruent, the six smaller equilateral triangles are all congruent and the six rhombi are also congruent. Since the vertices of the equilateral triangles occur at the midpoints of the sides of the hexagon, it follows that the side length of each rhombus and each smaller equilateral triangle is 10 cm. Thus the side length for the two larger equilateral triangles is $a = 30$ cm. Therefore, the area $A_T$ for each of
the larger equilateral triangles is
\[ A_T = \frac{\sqrt{3}}{4} a^2 = \frac{\sqrt{3}}{4} (30)^2 = 225\sqrt{3} \text{ cm}^2. \]

c) The circle is the inscribed circle for each of the two equilateral triangles with side length \( a = 30 \text{ cm}. \) Therefore, its radius \( r \) is given by \( r = \frac{\sqrt{3}}{6} a = 5\sqrt{3} \) cm. Hence, the area \( A_C \) of the circle is
\[ A_C = \pi r^2 = \pi (5\sqrt{3})^2 = 75\pi \text{ cm}^2. \]

**CC272.** A sum-palindrome number (SPN) is a number that, when there are an even number of digits, the first half of the digits sums to the same total as the second half of the digits, and when odd, the digits to the left of the central digit sum to the same total as the digits to the right of the central digit. A product-palindrome number (PPN) is like a sum-palindrome, except the products of the digits are involved, not the sums.

a) How many three-digit SPNs are there?

b) The two SPNs 1203 and 4022 sum to 5225, which is itself a SPN. Is it true that, for any two four-digit SPNs less than 5000, their sum is also a SPN?

c) How many four-digit non-zero PPNs are there?

*Originally Question 2 from the 2011 University of Otago Junior Mathematics Competition.*

We received 2 solutions, one of which was correct and complete. We present the solution by Ieko Dimitrić.

a) A three-digit SPN is of the form \( aba \) where \( b \), the digit of tens, can be any digit 0 to 9 (thus ten choices) and digit \( a \) of hundreds cannot be zero but can be any number 1 to 9, thus 9 choices. Then there are \( 10 \cdot 9 = 90 \) ways to choose \( a \) and \( b \) independently, and hence there are 90 three-digit SPNs.

b) The statement is not true, in particular in some situations that involve “carrying over” of units to a higher-value place, such as in examples
\[ 3544 + 4316 = 7860, \quad 2222 + 1964 = 4186, \quad 4114 + 2727 = 6841, \quad 1991 + 1991 = 3982. \]

c) First, we determine the number of four-digit PPNs in which 0 does not appear among the digits. We do the counting according to the number of distinct digits used in the representation of a PPN.

(1) There are nine PPNs if all the digits are equal.

(2) If exactly two distinct digits \( a \) and \( b \) are used, then the same pair is used in the first half of the number as in the second half for the following four
possibilities for each choice of \( a \) and \( b \): \( abab, abba, baab, baba \). Since a pair of distinct digits can be chosen from the set of nine in \( \binom{9}{2} = 36 \) ways, each choice producing four PPNs, the total number produced this way is \( 36 \cdot 4 = 144 \).

(3) If exactly three digits \( a, b \) and \( c \) are used, for such a number to be a PPN the product of two of them equals the square of the third, e. g. \( a \cdot b = c^2 \), whereas a pair \( a, b \) appears in one half of the number and digit \( c \), repeated twice, in the other half. This happens for the following four products of pairs

\[
1 \cdot 4 = 2 \cdot 2, \quad 1 \cdot 9 = 3 \cdot 3, \quad 2 \cdot 8 = 4 \cdot 4, \quad 4 \cdot 9 = 6 \cdot 6,
\]

each of the cases producing four PPNs, \( abcc, bacc, ccab, ccba \), therefore there are sixteen PPNs obtained this way.

(4) If four distinct digits are used for a PPN with two different sets of pairs \( a, b \) and \( c, d \) in each half, we must have \( ab = cd \). Since all four digits are non-zero and different, it is easily seen that \( ac \neq bd \) and \( ad \neq bc \), which means that the numbers from different pairs cannot be swapped and combined in the same half of the number and would have to stay within the original pair. This situation happens for the following five products of pairs:

\[
1 \cdot 6 = 2 \cdot 3, \quad 1 \cdot 8 = 2 \cdot 4, \quad 2 \cdot 6 = 3 \cdot 4, \quad 2 \cdot 9 = 3 \cdot 6, \quad 3 \cdot 8 = 4 \cdot 6.
\]

Each of them gives rise to eight PPNs: \( abcd, abdc, bacd, badc, cdab, cdba, dcab, dcba \). Thus, there are \( 8 \cdot 5 = 40 \) PPNs obtained this way.

Adding up the numbers for all the possibilities we get \( 9 + 144 + 16 + 40 = 209 \) PPNs in which the product of digits in each half of the number is the same and 0 is not one of the digits.

**CC273.** Kakuro is the name of a number puzzle where you place numbers from 1 to 9 into empty boxes. There are three rules in a Kakuro puzzle: only numbers from 1 to 9 may be used, no number is allowed in any line (across or down) more than once, the numbers must add up to the totals shown at the top and the left. The left figure in the diagram shows a small finished Kakuro puzzle. Solve the Kakuro puzzle on the right in the diagram. Is your solution unique?

![Kakuro Puzzles](image)

*Originally Question 1 from the 2008 University of Otago Junior Mathematics Competition.*

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We received one submission. We present the solution by Ivko Dimitrić.

The only way (up to the order) to write 16 as a sum of two distinct positive integers which are less than ten is $16 = 7 + 9 = 9 + 7$. Thus, these are the choices for the Sum-16 row and Sum-16 column.

Assume the Sum-16 row (the first row) is $[9 \mid 7]$. The Sum-23 column is then $[7 \mid a \mid b]^T$, where subscript $T$ denotes transpose, i.e. the triple of numbers should be seen in a vertical column. Then $a + b = 16$, where $a, b$ are distinct integers from 1 to 9, so one of them must be 7 again, which cannot happen, since we already have one 7 in the top box of that column. Hence, our assumption cannot stand and the first row is then $[7 \mid 9]$, whereupon the Sum-23 column takes the form $[9 \mid a \mid b]^T$, with $a + b = 14$.

Let the Sum-24 column be $[7 \mid c \mid d]^T$ where $c + d = 17$ so that one of the numbers $c, d$ is 9 the other one 8, in particular $c$ is either 8 or 9. Since 9 already appears in the Sum-23 column, according to the rules, $a + b = 14$ is possible only if it is the sum of 8 and 6. Assume first that the Sum-23 column is $[9 \mid 8 \mid 6]^T$. We show that this is not possible.

If the Sum-16 column (the last column) is $[7 \mid 9]^T$, then the Sum-29 row has the form $[f \mid d \mid 6 \mid 9]$, with $f + d = 14 = 9 + 5 = 8 + 6$, so one term in the sum would have to be 6 or 9, clashing with the same number already in the last row. If the Sum-16 column is $[9 \mid 7]^T$, then the Sum-30 row would appear as $[e \mid c \mid 8 \mid 9]$, so the number $c$ (which, from the above, is either 8 or 9) would clash with one of the last two numbers. Thus, both possibilities for the last column contradict the choice of the Sum-23 column as $[9 \mid 8 \mid 6]^T$, which means that that column would have to be changed to $[9 \mid 6 \mid 8]^T$ instead. Then, since $c + d = 17 = 9 + 8$ in the Sum-24 column, we have $c = 8$ and $d = 9$ and that column is completed as $[7 \mid 8 \mid 9]^T$ so that the last column could be only $[9 \mid 7]^T$, to avoid having two 9s in the last row. Then the remaining cell in the Sum-30 row is filled with $e = 7$ and the remaining cell in the Sum-29 row is filled with $f = 5$ to make the required sums, including the remaining Sum-12 column.

Since every choice at every step is forced upon by previous choices or contradictions resulting from alternative possibilities, every number is uniquely determined, i.e. there is only one solution to the puzzle, namely

![Image of the puzzle grid](image_url)
CC274. In his office, Shaquille had nine ping pong balls which he used for therapeutic recovery by throwing them into the waste basket at slack times. Each time he threw the nine balls, some of them would land in the basket, with the rest of them landing on the floor.

a) If the balls are identical, how many different results could there be?

b) Suppose now that the balls are numbered 1 to 9. How many different results could there be now? (For example one possible result is for balls 1 to 4 to land in the basket, with 5 to 9 on the floor.)

c) Suppose instead that the balls are not numbered, but five are coloured yellow and four blue. Now how many different results could there be? (For example one possible result is for two yellow balls and three blue balls to land in the basket, and the rest to land on the floor.)

d) One day another basket appeared in the office. So now Shaquille had a choice of baskets to aim at. How did this change the answers to (a), (b), and (c)?

e) Now suppose that every time he threw the balls at the two baskets, each basket received at least two balls. How would this change the answers to (a), (b), and (c)?

Originally Question 5 from the 2008 University of Otago Junior Mathematics Competition.

We received one solution to this problem. We present the solution by Ieko Dimitrić, modified by the editor.

In each case, the number and the selection of the balls that land on the floor is uniquely determined by the number and the selection of the balls that land in the basket(s), so it suffices to count the number of balls that land in the basket(s) to get the number of different results.

a) Any number \( k = 0, 1, \ldots, 9 \) of balls can land in the basket. Therefore there are 10 different results.

b) A numbered ball is either in or out of the basket. Therefore each numbered ball has 2 possible states. As there are 9 balls, there are \( 2^9 = 512 \) different results.

c) Since any number \( y = 0, 1, \ldots, 5 \) of yellow balls and any number \( b = 0, 1, \ldots, 4 \) of blue balls can independently land in the basket we have \( 6 \cdot 5 = 30 \) different results.

d) Let \( B_L \) and \( B_R \) refer to the “left basket” and “right basket,” respectively.

To answer a), for any number \( k = 0, 1, \ldots, 9 \) of balls that land in \( B_L \) there are \( 10 - k \) choices of balls that may land in \( B_R \). Therefore there are \( \sum_{k=0}^{9} (10 - k) = 55 \) different results.

To answer b), a numbered ball is either in \( B_L \), in \( B_R \), or not in either. Therefore each numbered ball has 3 possible states. As there are 9 balls, there are \( 3^9 = 19,683 \) different results.
To answer c), for any number \( y = 0, 1, \ldots, 5 \) of yellow balls that land in \( B_L \) there are \( 6 - y \) choices of yellow balls that may land in \( B_R \). Therefore there are \( \sum_{y=0}^{5} (6 - y) = 21 \) ways to distribute the 5 yellow balls. Likewise, for any number \( b = 0, 1, \ldots, 4 \) of blue balls that land in \( B_L \) there are \( 5 - b \) choices of blue balls that may land in \( B_R \). Therefore there are \( \sum_{b=0}^{4} (5 - b) = 15 \) ways to distribute the 4 blue balls. Therefore there are \( 21 \cdot 15 = 315 \) different results.

e) To answer a) for any number \( k = 2, 3, \ldots, 7 \) of balls that land in \( B_L \) there are \( 8 - k \) choices of balls that may land in \( B_R \). Therefore there are \( \sum_{k=2}^{7} (8 - k) = 21 \) different results.

To answer (b), choose \( n \geq 4 \) balls out of nine in \( \binom{9}{n} \) ways to be distributed in baskets \( B_L \) and \( B_R \). Given the \( n \) chosen balls, \( k \) land in basket \( B_L \) and \( n - k \) land in basket \( B_R \). Therefore \( 2 \leq k \leq n - 2 \). Thus, the number of results is

\[
\sum_{n=4}^{9} \sum_{k=2}^{n-2} \binom{n}{k} \binom{n-2}{k} = \sum_{n=4}^{9} \frac{9}{n} \left[ \sum_{k=0}^{n} \binom{n}{k} - \binom{n}{0} - \binom{n}{1} - \binom{n}{n-1} - \binom{n}{n} \right] = \sum_{n=4}^{9} \frac{9}{n} [2^n - 2 - 2n].
\]

We complete the sum so that it starts from \( n = 0 \) by adding and subtracting terms corresponding to \( n = 0, 1, 2, 3 \), which are equal to \(-1, -18, -72, 0\), respectively. The above sum becomes

\[
\sum_{n=0}^{9} \frac{9}{n} [2^n - 2 - 2n] + 91 = \sum_{n=0}^{9} \frac{9}{n} 2^n - 2 \sum_{n=0}^{9} \frac{9}{n} n + 91 = (2 + 1)^9 - 2(1 + 1)^8 - 2 \cdot 9 \cdot 2^8 + 91 = 3^9 - 11 \cdot 2^9 + 91 = 14,142.
\]

Therefore there are 14,142 different results.

To answer c), we count the number of results in which either or both of the baskets end up with less than two balls and subtract that number from 315 (the number of results we found in part (d) without any restrictions).

Let the sets \( (L_Y, L_B) \) and \( (R_Y, R_B) \) correspond to the number of yellow and blue balls that land in \( B_L \) and \( B_R \), respectively. If \( B_L \) ends up with less than 2 balls then the set \( (L_Y, L_B) \in \{(0,0), (1,0), (0,1)\} \). We consider these three cases:

1. If \( (L_Y, L_B) = (0,0) \), then there are 6 choices of yellow balls and 5 choices of blue balls landing in \( B_R \). Totaling \( 6 \cdot 5 = 30 \) different results.

2. If \( (L_Y, L_B) = (1,0) \), then there are 5 choices of yellow balls and 5 choices of blue balls landing in \( B_R \). Totaling \( 5 \cdot 5 = 25 \) different results.

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3. If \((L_y, L_b) = (0, 1)\), then there are 6 choices of yellow balls and 4 choices of blue balls landing in \(B_r\). Totalling \(6 \cdot 4 = 24\) different results.

A total of 79 results. If the situation is reversed and the basket \(B_R\) receives less than two balls, we will also have 79 results. In their sum the number of results where both baskets got less than two balls is counted twice. That number is \(3 \cdot 3 = 9\), since there are three possibilities for each basket as listed above. The total number of results in which at least one basket has less than two balls is \(2 \cdot 79 - 9 = 149\). The number of results in which both baskets received at least two balls is therefore \(315 - 149 = 166\).

**CC275.** The local sailing club is planning its annual race. By tradition the boats always start at \(P\), sailing due North for a distance of \(a\) km until they reach \(Q\). They then turn and sail a distance of \(x\) km to \(R\) (which is due East of \(P\)). Next they turn and sail a distance of \(y\) km to \(S\) (which is South of \(P\)) before finally sailing due North for a distance of \(b\) km until the finish line, which is back at the starting point \(P\) (see the diagram). Bernie, the Club Commander, makes four extra rules for this year’s race:

- \(a + b = 40\),
- \(x + y = 50\),
- \(a < b\),
- the four lengths \((a, b, x, y)\) must each be a whole number of kilometres. (Bernie doesn’t like decimals.)

a) Find four numbers \((a, b, x, y)\) which satisfy Bernie’s four rules.

b) Are there four different numbers \((a, b, x, y)\), which also satisfy Bernie’s four rules apart from the four numbers you found in part (a)? Explain.

*Originally Question 3 from the 2008 University of Otago Junior Mathematics Competition.*

We received four solutions to this problem, all of which were correct. We present the composite solution of Konstantine Zelator, Digby Smith, and Ivko Dimitrić, modified by the editor.

We solve a) and b) by determining all solutions to Bernie’s problem. From the right angle triangles \(PQR\) and \(PRS\) we have that

\[
(PR)^2 = x^2 - a^2 = y^2 - b^2
\]

Since \(a + b = 40 \Rightarrow b = 40 - a\). Since \(x + y = 50 \Rightarrow y = 50 - x\). Through substitution we see that

\[
x^2 - a^2 = (50 - x)^2 - (40 - a)^2,
\]
which simplifies to \( x = 9 + \frac{4a}{5} \). Since \( x \) is a whole number it follows that \( a \) is divisible by 5. Note that, since \( a < b \), \( a < \frac{a+b}{2} = 20 \). Therefore \( a \) is a whole number that is less than 20 and divisible by 5. Assuming \( a > 0 \) the possible values of \( a \) are 5, 10, and 15. Below we list these 3 cases of \( a \):

\[
\begin{align*}
    a = 5 & \implies b = 35, x = 13, y = 37. \\
    a = 10 & \implies b = 30, x = 17, y = 33. \\
    a = 15 & \implies b = 25, x = 21, y = 29.
\end{align*}
\]

The three quadruples that solve Bernie’s problem are therefore

\((5, 35, 13, 37), (10, 30, 17, 33)\) and \((15, 25, 21, 29)\).

Note that if \( a \geq 0 \) then \((0, 40, 9, 41)\) is also a solution to Bernie’s problem.

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**Square Garlands**

Using construction paper, a string and some glue, make 5 garlands consisting of 5 squares each (for a sturdy garland, glue squares to both sides of the string).

Using these garlands, can you completely cover the figure shown above?

*Puzzle by Nikolai Avilov.*