THE OLYMPIAD CORNER

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The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by October 1, 2018.

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OC381. The integers 1, 2, 3, . . . , 2016 are written on a board. You can choose any two numbers on the board and replace them with one copy of their average. For example, you can replace 1 and 2 with 1.5, or you can replace 1 and 3 with a second copy of 2. After 2015 replacements of this kind, the board will have only one number left on it.

a) Prove that there is a sequence of replacements that will make the final number equal to 2.

b) Prove that there is a sequence of replacements that will make the final number equal to 1000.

OC382. There are $n > 1$ cities in a country and some pairs of cities are connected by two-way non-stop flights. Moreover, every two cities are connected by a unique route (possibly with stopovers). A mayor of every city $X$ counted the number of labelings of the cities from 1 to $n$ so that every route beginning with $X$ has the rest of the cities on that route occurring in ascending order. Every mayor, except one, noticed that the resulting number of their labelling is a multiple of 2016. Prove that last mayor’s number of labelings is also a multiple of 2016.

OC383. Let $ABC$ be a triangle. Let $r$ and $s$ be the angle bisectors of $\angle ABC$ and $\angle BCA$, respectively. The points $E$ in $r$ and $D$ in $s$ are such that $AD \parallel BE$ and $AE \parallel CD$. The lines $BD$ and $CE$ cut each other at $F$. The point $I$ is the incenter of $ABC$. Show that if $A, F$ and $I$ are collinear, then $AB = AC$.

OC384. Solve the equation $xyz + yzt + xzt + xyt = xyzt + 3$ over the set of natural numbers.

OC385. A subset $S \subset \{0, 1, 2, \cdots , 2000\}$ satisfies $|S| = 401$. Prove that there exists a positive integer $n$ such that there are at least 70 positive integers $x$ such that $x, x+n \in S$.
**OC381.** On a écrit les entiers 1, 2, 3, ..., 2016 au tableau. Une action consiste à enlever n’importe quels deux nombres du tableau et ajouter leur moyenne à la liste, ce qui a pour effet de raccourcir la liste d’un terme. Par exemple, vous pouvez enlever les nombres 1 et 2 et ajouter le nombre 1,5 ou vous pouvez enlever les nombres 1 et 3 et ajouter le nombre 2. Après 2015 actions, il n’y aura plus qu’un seul nombre dans la liste.

a) Démontrer qu’il existe une suite d’actions telle qu’à la fin, le dernier nombre soit 2.

b) Démontrer qu’il existe une suite d’actions telle qu’à la fin, le dernier nombre soit 1000.

**OC382.** Il y a n (n > 1) villes dans un pays et certaines d’entre elles sont reliées par des vols directs aller-retour. Il y a exactement un lien aérien entre chaque paire de villes, possiblement avec escales et changements d’avion. Le maire de chaque ville X a compté le nombre d’étiquetages des villes de 1 à n de manière que sur chaque route à partir de X, les autres villes de cette route soient placées en ordre ascendant. Chaque maire, sauf un, remarque que le nombre d’étiquetages comptés est un multiple de 2016. Démontrer que le nombre d’étiquetages de ce dernier maire est aussi un multiple de 2016.

**OC383.** Soit un triangle ABC. Soit r et s les bissectrices des angles respectifs ABC et BCA. On considère les points E sur r et D sur s tels que AD∥BE et AE∥CD. Soit F le point d’intersection des droites BD et CE. Soit I le centre du cercle inscrit dans le triangle ABC. Démontrer que si A, F et I sont alignés, alors AB = AC.

**OC384.** Résoudre l’équation $xyz + yz + xz + y = xyz + 3$ dans l’ensemble des nombres naturels.

**OC385.** Un sous-ensemble S de $\{0, 1, 2, \cdots, 2000\}$ vérifie $|S| = 401$. Démontrer qu’il existe un entier strictement positif n de manière qu’il existe au moins 70 entiers strictement positifs $x$ tels que $x, x + n \in S$. 

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OLYMPIAD SOLUTIONS


OC321. Solve in positive integers

\[ x^y y^x = (x + y)^z. \]

Originally 2015 Kazakhstan National Olympiad.

We received 3 submissions of which 1 was correct and complete. We present the solution by Steven Chow.

If \( x = 1 \), then \( x^y y^x = (x + y)^z \Rightarrow y = (y + 1)^z \geq y + 1 \) which is a contradiction.

Therefore \( x \geq 2 \) and similarly, \( y \geq 2 \).

If \( p \) is a prime such that \( p \mid x \), then \( p \mid x^y y^x = (x + y)^z \), so \( p \mid x + y \) and \( p \mid y \).

If \( p \) is a prime such that \( p \mid y \), then similarly, \( p \mid x + y \) and \( p \mid x \).

Therefore \( x, y, \) and \( x + y \) have the same primes in their prime-power factorizations.

Let \( \prod_{j=1}^{k} p_j^{\alpha_j} = x \) and \( \prod_{j=1}^{k} p_j^{\beta_j} = y \) be the prime-power factorizations of \( x \) and \( y \).

Since \( x^y y^x = (x + y)^z \), therefore \( z \mid \alpha_j y + \beta_j x \) for all \( j \) and

\[
\prod_{j=1}^{k} p_j^{\alpha_j} + \prod_{j=1}^{k} p_j^{\beta_j} = x + y = (x^y y^x)^{\frac{1}{2}} = \prod_{j=1}^{k} p_j^{\alpha_j y + \beta_j x}. 
\]

If there exists \( i \) such that \( \alpha_i \neq \beta_i \), then WLOG, assume that \( \alpha_i < \beta_i \), so

\[
p_i^{\alpha_i} \mid \prod_{j=1}^{k} p_j^{\alpha_j} + \prod_{j=1}^{k} p_j^{\beta_j},
\]

so

\[
\alpha_i = \frac{\alpha_i y + \beta_i x}{z} > \frac{\alpha_i (x + y)}{z} \Rightarrow z > x + y \Rightarrow (x + y)^z > x^y y^x
\]

from the Binomial Theorem, which is a contradiction.

Therefore for all \( j \), \( \alpha_j = \beta_j \).

Therefore \( x = y \), so \( x^y y^x = (x + y)^z \iff x^{2x} = 2^z x^z \).

If there exists a prime \( p \neq 2 \) such that \( p \mid x \), then since \( x^{2x} = 2^z x^z \), \( 2x = z \), so

\( 1 = 2^{2x} \Rightarrow x = 0 \), which is a contradiction.

Let \( x_1 \geq 1 \) be the integer such that \( 2^{x_1} = x \).
Therefore

\[ x^{2x} = 2^x x^2 \iff (2^{x_1})^{2(2^{x_1})} = 2^z (2^{x_1})^z \]

\[ \iff x_1 2^{x_1+1} = (x_1 + 1) z \iff z = \frac{x_1 2^{x_1+1}}{x_1 + 1}. \]

Since \( \gcd(x_1, x_1 + 1) = 1 \), the last number is a positive integer if and only if

\[ x_1 = 2^{x_2} - 1 \]

for some integer \( x_2 \geq 1 \).

Hence, the solution is \( x = y = 2^{j-1} \) and \( z = (2^j - 1) 2^{2j-j} \) for any integer \( j \geq 1 \).

Editor's Comments. The other 2 solvers misinterpreted the problem and found the positive integer solutions to the equation

\[ x^y y^z = (x + y)^z. \]

This is an easier problem and we will leave it as an exercise to the reader.

**OC322.** Let \( a, b, c \in \mathbb{R}^+ \) such that \( abc = 1 \). Prove that

\[ a^2 b + b^2 c + c^2 a \geq \sqrt{(a + b + c)(ab + bc + ca)}. \]

*Originally 2015 Macedonia National Olympiad Problem 2.*

*We received 6 solutions. We present the solution by Titu Zvonaru.*

Using the AM-GM Inequality and \( abc = 1 \), we have

\[ a^2 b + a^2 b + b^2 c \geq 3 \sqrt[3]{a^4 b^2 c} = 3ab \]

\[ b^2 c + b^2 c + c^2 a \geq 3 \sqrt[3]{b^4 c^2 a} = 3bc \]

\[ c^2 a + c^2 a + a^2 b \geq 3 \sqrt[3]{c^4 a^2 b} = 3ca. \]

Adding these three inequalities, we get

\[ a^2 b + b^2 c + c^2 a \geq ab + bc + ca. \tag{1} \]

Similarly,

\[ a^2 b + b^2 c + b^2 c \geq 3 \sqrt[3]{a^3 b^2 c^2} = 3b \]

\[ b^2 c + c^2 a + c^2 a \geq 3 \sqrt[3]{b^4 c^2 a^2} = 3c \]

\[ c^2 a + a^2 b + a^2 b \geq 3 \sqrt[3]{a^4 b^2 c} = 3a, \]

hence

\[ a^2 b + b^2 c + c^2 a \geq a + b + c. \tag{2} \]

By (1) and (2), we get the desired inequality. The equality holds if and only if \( a = b = c = 1 \).
Editor’s comments. Dan Daniel used the same approach, but using the clever substitution $a = \frac{z}{y}$, $b = \frac{y}{z}$, $c = \frac{x}{z}$, where $x, y, z > 0$. Indeed, then the given inequality becomes

$$\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} \geq \sqrt{\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)\left(\frac{x}{z} + \frac{y}{x} + \frac{z}{y}\right)},$$

which is equivalent to

$$(x^3 + y^3 + z^3)^2 \geq (x^2y + y^2z + z^2x)(x^2z + y^2x + z^2y).$$

Now, it is sufficient to prove that $x^3 + y^3 + z^3 \geq x^2y + y^2z + z^2x$ and $x^3 + y^3 + z^3 \geq x^2z + y^2x + z^2y$. For that, the approach is the same as the one used by Titu Zvonaru in the solution above.

**OC323.** Let $ABC$ be a triangle. $M$, and $N$ points on $BC$, such that $BM = CN$, with $M$ in the interior of $BN$. Let $P$ and $Q$ be points in $AN$ and $AM$ respectively such that $\angle PMC = \angle MAB$, and $\angle QNB = \angle NAC$. Prove that $\angle QBC = \angle PCB$.

*Originally 2015 Spain Mathematical Olympiad Day 2 Problem 3.*

We received 3 correct solutions and will present 2 solutions.

**Solution 1,** by Mohammed Aassila.

Let $A'$ and $P'$ be the reflections of $A$ and $P$ with respect to the perpendicular bisector of $BC$, respectively. Let $\{X\} = NQ \cap A'B$ and $\{Y\} = NP' \cap AB$. Then it is easy (from symmetry) to see that $P' \in A'M$.

Since $AA'MN$ is an isosceles trapezoid, then $A$, $A'$, $M$, $N$ are concyclic. Since $\angle XNM = \angle NAC = \angle XA'M$, then $A'$, $X$, $M$, $N$ are concyclic. Since $\angle YNM = \angle NMP = \angle YAM$, then $A$, $Y$, $N$, $M$ are concyclic.

Therefore, $A$, $A'$, $M$, $N$, $X$, $Y$ are concyclic, so from Pascal’s theorem (applied to $AYNXA'M$), we conclude that

$$P' \in BQ \implies \angle QBC = \angle P'BC = \angle PCB.$$ 

**Solution 2,** by Andrea Fanchini.

We use barycentric coordinates and the usual Conway’s notations with reference to the triangle $ABC$.

Points $M$ and $N$ have coordinates

$$M(0 : a - t : t), \quad N(0 : t : a - t)$$

where $t$ is a parameter.
We now calculate the coordinates of point $P$. Recall that the oriented angle $\theta$ ($0 \leq \theta \leq \pi$) of the oriented lines $d_i \equiv p_i x + q_i y + r_i z = 0 (i = 1, 2)$, is given from

$$S_\theta = S \cot \theta = \frac{S_A(q_1 - r_1)(q_2 - r_2) + S_B(r_1 - p_1)(r_2 - p_2) + S_C(p_1 - q_1)(p_2 - q_2)}{\begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}$$

so $\angle MAB$ between the two lines $AB : z = 0$ and $AM : ty + (t - a)z = 0$ is

$$S_{\angle MAB} = \frac{ac^2 - S_B t}{t}.$$ 

Now the side $BC : x = 0$ forms an angle $\angle PMC = \angle MAB$ with the line $PM$. Therefore, the point at infinity is

$$PM_\infty \left( a^2 t : (S_B - S_C) t - ac^2 : ac^2 - 2S_B t \right).$$

Then the line that passes from $M$ and has $PM_\infty$ as the point at infinity is

$$MPM_\infty : (at^2 - 2S_B t + ac^2) x + at^2 y + at(t - a) z = 0.$$ 

The line $AN$ is given by $(t - a) y + tz = 0$. Thus, the point $P$ has coordinates

$$P = AN \cap MPM_\infty = (a^2 t (2t - a) : t(2S_B t - at^2 - ac^2) : (t - a) (at^2 - 2S_B t + ac^2))$$

Let us now calculate the coordinates of point $Q$. The $\angle NAC$ between the two lines $AN : (t - a) y + tz = 0$ and $AC : y = 0$ is

$$S_{\angle NAC} = \frac{ab^2 - S_C t}{t}.$$ 

Now the side $BC : x = 0$ forms an oriented angle $\pi - \angle QNB = \pi - \angle NAC$ with the line $QN$, but $S_{\pi - \angle QNB} = - S_{\angle NAC}$. Therefore the point at infinity of this line is

$$QN_\infty \left( a^2 t : ab^2 - 2S_C t : (S_C - S_B) t - ab^2 \right).$$

Then the line that passes from $N$ and has the infinite point $QN_\infty$ is

$$NQN_\infty : (2S_C t - at^2 - ab^2) x + at(a - t) y - at^2 z = 0.$$ 

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Therefore the point \( Q \) has coordinates
\[
Q = AM \cap NQN_\infty = (a^2 t (2t - a) : (t - a)(at^2 - 2S_C t + ab^2) : t(2S_C t - at^2 - ab^2))
\]

Finally, we will show that \( \angle QBC = \angle PCB \). The \( \angle QBC \) between the two lines \( BC : x = 0 \) and \( BQ : (2S_C t - at^2 - ab^2)x + a^2 (a - 2t)z = 0 \) is
\[
S_{\angle QBC} = \frac{a (t^2 - 2at + S_A + S_B + S_C)}{a - 2t}
\]
The \( \angle PCB \) between the two lines \( PC : (at^2 - 2S_B t + ac^2)x + a^2 (2t - a) y = 0 \) and \( BC : x = 0 \) is
\[
S_{\angle PCB} = \frac{a (t^2 - 2at + S_A + S_B + S_C)}{a - 2t}
\]
and we are done.

**OC324.** Given an integer \( n > 1 \) and its prime factorization \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \), define a function
\[
f(n) = \alpha_1 p_1^{\alpha_1 - 1} \alpha_2 p_2^{\alpha_2 - 1} \cdots \alpha_k p_k^{\alpha_k - 1}.
\]
Prove that there exist infinitely many integers \( n \) such that \( f(n) = f(n - 1) + 1 \).

*Originally 2015 Brazil National Olympiad Problem 3 Day 1.*

*No solutions received.*

**OC325.** Let \( S = \{1, 2, \ldots, n\} \), where \( n \geq 1 \). Each of the \( 2^n \) subsets of \( S \) is to be coloured red or blue. (The subset itself is assigned a colour and not its individual elements.) For any set \( T \subseteq S \), we then write \( f(T) \) for the number of subsets of \( T \) that are blue.

Determine the number of colourings that satisfy the following condition: for any subsets \( T_1 \) and \( T_2 \) of \( S \),
\[
f(T_1) f(T_2) = f(T_1 \cup T_2) f(T_1 \cap T_2).
\]

*Originally 2015 USAMO Day 1 Problem 3.*

*No solutions received.*

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