OLYMPIAD SOLUTIONS


OC321. Solve in positive integers

\[ x^y y^x = (x + y)^z. \]

Originally 2015 Kazakhstan National Olympiad.

We received 3 submissions of which 1 was correct and complete. We present the solution by Steven Chow.

If \( x = 1 \), then \( x^y y^x = (x + y)^z \) \( \implies \) \( y = (y + 1)^z \geq y + 1 \) which is a contradiction. Therefore \( x \geq 2 \) and similarly, \( y \geq 2 \).

If \( p \) is a prime such that \( p \mid x \), then \( p \mid x^y y^x = (x + y)^z \), so \( p \mid x + y \) and \( p \mid y \). If \( p \) is a prime such that \( p \mid y \), then similarly, \( p \mid x + y \) and \( p \mid x \).

Therefore \( x, y, \) and \( x+y \) have the same primes in their prime-power factorizations.

Let \( \prod_{j=1}^{k} p_j^{\alpha_j} = x \) and \( \prod_{j=1}^{k} p_j^{\beta_j} = y \) be the prime-power factorizations of \( x \) and \( y \).

Since \( x^y y^x = (x + y)^z \), therefore \( z \mid \alpha_jy + \beta_jx \) for all \( j \) and

\[
\prod_{j=1}^{k} p_j^{\alpha_j} + \prod_{j=1}^{k} p_j^{\beta_j} = x + y = (x^y y^x)^{\frac{1}{z}} = \prod_{j=1}^{k} p_j^{\frac{\alpha_j y + \beta_j x}{z}}.
\]

If there exists \( i \) such that \( \alpha_i \neq \beta_i \), then WLOG, assume that \( \alpha_i < \beta_i \), so

\[
p_{i}^{\alpha_i} \mid \prod_{j=1}^{k} p_j^{\alpha_j} + \prod_{j=1}^{k} p_j^{\beta_j},
\]

so

\[
\alpha_i = \frac{\alpha_i y + \beta_i x}{z} > \frac{\alpha_i (x + y)}{z} \implies z > x + y \implies (x + y)^z > x^y y^x
\]

from the Binomial Theorem, which is a contradiction.

Therefore for all \( j \), \( \alpha_j = \beta_j \).

Therefore \( x = y \), so \( x^y y^x = (x + y)^z \iff x^{2x} = 2^z x^z \).

If there exists a prime \( p \neq 2 \) such that \( p \mid x \), then since \( x^{2x} = 2^z x^z \), \( 2x = z \), so \( 1 = 2^{2x} \implies x = 0 \), which is a contradiction.

Let \( x_1 \geq 1 \) be the integer such that \( 2^{x_1} = x \).
Therefore
\[ x^{2z} = 2^z x^z \iff (2^{x_1})^{2(2^{x_1})} = 2^z (2^{x_1})^z \]
\[ \iff x_1 2^{x_1+1} = (x_1 + 1) z \iff z = \frac{x_1 2^{x_1+1}}{x_1 + 1} . \]

Since \( \gcd(x_1,x_1+1) = 1 \), the last number is a positive integer if and only if \( x_1 = 2^{x_2} - 1 \) for some integer \( x_2 \geq 1 \). Then
\[ z = (2^{x_2} - 1) 2^{2^{x_2} - x_2} . \]
Hence, the solution is \( x = y = 2^{x_2} - 1 \) and \( z = (2^{x_2} - 1) 2^{2^{x_2} - x_2} \) for any integer \( j \geq 1 \).

Editor’s Comments. The other 2 solvers misinterpreted the problem and found the positive integer solutions to the equation
\[ xy^x = (x + y)^x . \]
This is an easier problem and we will leave it as an exercise to the reader.

**OC322.** Let \( a, b, c \in \mathbb{R}^+ \) such that \( abc = 1 \). Prove that
\[ a^2 b + b^2 c + c^2 a \geq \sqrt{(a + b + c)(ab + bc + ca)} . \]

*Originally 2015 Macedonia National Olympiad Problem 2.*

*We received 6 solutions. We present the solution by Titu Zvonaru.*

Using the AM-GM Inequality and \( abc = 1 \), we have
\[ a^2 b + a^2 b + b^2 c \geq 3 \sqrt[3]{a^4 b^4 c} = 3ab \]
\[ b^2 c + b^2 c + c^2 a \geq 3 \sqrt[3]{b^4 c^4 a} = 3bc \]
\[ c^2 a + c^2 a + a^2 b \geq 3 \sqrt[3]{c^4 a^4 b} = 3ca . \]

Adding these three inequalities, we get
\[ a^2 b + b^2 c + c^2 a \geq ab + bc + ca . \] (1)

Similarly,
\[ a^2 b + b^2 c + b^2 c \geq 3 \sqrt[3]{a^3 b^3 c^2} = 3b \]
\[ b^2 c + c^2 a + c^2 a \geq 3 \sqrt[3]{b^3 c^3 a^2} = 3c \]
\[ c^2 a + a^2 b + a^2 b \geq 3 \sqrt[3]{c^3 a^3 b^2} = 3a , \]

hence
\[ a^2 b + b^2 c + c^2 a \geq a + b + c . \] (2)

By (1) and (2), we get the desired inequality. The equality holds if and only if \( a = b = c = 1 \).

_Crux Mathematicorum, Vol. 44(5), May 2018_
Editor’s comments. Dan Daniel used the same approach, but using the clever substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$, where $x, y, z > 0$. Indeed, then the given inequality becomes

$$\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} \geq \sqrt{\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)\left(\frac{x}{z} + \frac{y}{x} + \frac{z}{y}\right)},$$

which is equivalent to

$$(x^3 + y^3 + z^3)^2 \geq (x^2y + y^2z + z^2x)(x^2z + y^2x + z^2y).$$

Now, it is sufficient to prove that $x^3 + y^3 + z^3 \geq x^2y + y^2z + z^2x$ and $x^3 + y^3 + z^3 \geq x^2z + y^2x + z^2y$. For that, the approach is the same as the one used by Titu Zvonaru in the solution above.

**OC323.** Let $ABC$ be a triangle. $M$, and $N$ points on $BC$, such that $BM = CN$, with $M$ in the interior of $BN$. Let $P$ and $Q$ be points in $AN$ and $AM$ respectively such that $\angle PMC = \angle MAB$, and $\angle QNB = \angle NAC$. Prove that $\angle QBC = \angle PCB$.

*Originally 2015 Spain Mathematical Olympiad Day 2 Problem 3.*

*We received 3 correct solutions and will present 2 solutions.*

**Solution 1, by Mohammed Aassila.**

Let $A'$ and $P'$ be the reflections of $A$ and $P$ with respect to the perpendicular bisector of $BC$, respectively. Let $\{X\} = NQ \cap A'B$ and $\{Y\} = NP' \cap AB$. Then it is easy (from symmetry) to see that $P' \in A'M$.

Since $AA'MN$ is an isosceles trapezoid, then $A$, $A'$, $M$, $N$ are concyclic. Since $\angle XNM = \angle NAC = \angle XA'M$, then $A'$, $X$, $M$, $N$ are concyclic. Since $\angle YNM = \angle NMP = \angle YAM$, then $A$, $Y$, $N$, $M$ are concyclic.

Therefore, $A$, $A'$, $M$, $N$, $X$, $Y$ are concyclic, so from Pascal’s theorem (applied to $AYNXA'M$), we conclude that

$$P' \in BQ \implies \angle QBC = \angle P'BC = \angle PCB.$$  

**Solution 2, by Andrea Fanchini.**

We use barycentric coordinates and the usual Conway’s notations with reference to the triangle $ABC$.

Points $M$ and $N$ have coordinates

$$M(0 : a - t : t), \quad N(0 : t : a - t)$$

where $t$ is a parameter.
We now calculate the coordinates of point \( P \). Recall that the oriented angle \( \theta \) 
\((0 \leq \theta \leq \pi)\) of the oriented lines \( d_i \equiv p_i x + q_i y + r_i z = 0 \) \((i = 1, 2)\), is given from

\[
S_\theta = S \cot \theta = \frac{S_A(q_1 - r_1)(q_2 - r_2) + S_B(r_1 - p_1)(r_2 - p_2) + S_C(p_1 - q_1)(p_2 - q_2)}{\begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}
\]

so \( \angle MAB \) between the two lines \( AB : z = 0 \) and \( AM : ty + (t-a)z = 0 \) is

\[
S_{\angle MAB} = \frac{ac^2 - S_B t}{t}.
\]

Now the side \( BC : x = 0 \) forms an oriented angle \( \pi - \angle QNB = \pi - \angle NAC \) with the line \( QN \), but \( S_{\pi - \angle QNB} = -S_{\angle NAC} \). Therefore the point at infinity of this line is

\[
QN_\infty \left( a^2 t : (S_B - S_C)t - ac^2 : ab^2 - 2S_B t \right).
\]

Then the line that passes from \( N \) and has the infinite point \( QN_\infty \) is

\[
NQN_\infty : (2S_C t - at^2 - ab^2)x + at(a-t)y - at^2 z = 0.
\]
Therefore the point $Q$ has coordinates

$$Q = AM \cap NQN_\infty = (a^2 t(2t - a) : (t - a)(at^2 - 2SCt + ab^2) : t(2SCt - at^2 - ab^2))$$

Finally, we will show that $\angle QBC = \angle PCB$. The $\angle QBC$ between the two lines $BC : x = 0$ and $BQ : (2SCt - at^2 - ab^2)x + a^2(a - 2t)z = 0$ is

$$S_{\angle QBC} = \frac{a \left( t^2 - 2at + S_A + S_B + S_C \right)}{a - 2t}$$

The $\angle PCB$ between the two lines $PC : (at^2 - 2SBt + ac^2)x + a^2(2t - a)y = 0$ and $BC : x = 0$ is

$$S_{\angle PCB} = \frac{a \left( t^2 - 2at + S_A + S_B + S_C \right)}{a - 2t}$$

and we are done.

**OC324.** Given an integer $n > 1$ and its prime factorization $n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, define a function

$$f(n) = \alpha_1 p_1^{\alpha_1 - 1} \alpha_2 p_2^{\alpha_2 - 1} \cdots \alpha_k p_k^{\alpha_k - 1}.$$ 

Prove that there exist infinitely many integers $n$ such that $f(n) = f(n - 1) + 1$.

*Originally 2015 Brazil National Olympiad Problem 3 Day 1.*

*No solutions received.*

**OC325.** Let $S = \{1, 2, \ldots, n\}$, where $n \geq 1$. Each of the $2^n$ subsets of $S$ is to be coloured red or blue. (The subset itself is assigned a colour and not its individual elements.) For any set $T \subseteq S$, we then write $f(T)$ for the number of subsets of $T$ that are blue.

Determine the number of colourings that satisfy the following condition: for any subsets $T_1$ and $T_2$ of $S$,

$$f(T_1) f(T_2) = f(T_1 \cup T_2) f(T_1 \cap T_2).$$

*Originally 2015 USAMO Day 1 Problem 3.*

*No solutions received.*