THE HONSBERGER CORNER


H1. Across the bottom of rectangular wine rack $PQRS$, there is room for more than three bottles ($A,B,C$) but not enough for a fourth bottle (see the figure). Naturally, bottles $A$ and $C$ are laid against the sides of the rack and a second layer, consisting of just two bottles $D$ and $E$, holds $B$ in place somewhere between $A$ and $C$. Now we can lay in a third row of three bottles ($F,G,H$), with $F$ and $H$ resting against the sides of the rack. Then a fourth layer is held to just two bottles $I$ and $J$, but a fifth layer can accommodate three bottles ($K,L,M$).

If the bottles are all the same size, prove that, whatever the spacing of ($A,B,C$) in the bottom layer, the fifth layer is always perfectly horizontal.

This is problem 44 from “Which Way Did the Bicycle Go? And Other Intriguing Mathematical Mysteries” by Joseph D. E. Konhauser, Dan Velleman and Stan Wagon (Cambridge University Press, 1996).

We received 5 submissions, of which 4 were correct and complete. We present the solution by Steven Chow, modified by the editor.

We will write $A$ for the center of the circle representing bottle $A$, and so on. Since all the bottles are the same size, the distance between the centers of the circles corresponding to any two tangent bottles is twice the common radius, and the point of tangency of any two tangent bottles is the midpoint of the segment connecting the centers.

Note that $\angle BAF = 90^\circ$ (bottles $B$, $A$ and $F$ are tangent to the rectangular rack). Since $A$, $B$, $F$ and $G$ are equidistant from $D$, the quadrilateral $ABFG$ is cyclic.

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and $D$ is the center of its circumcircle. It follows that $\angle BDF = 2\angle BAF = 180^\circ$, so $B$, $D$ and $F$ are collinear. Similarly, $B$, $E$ and $H$ are collinear.

Since all its sides are equal, $BEGD$ is a rhombus, as are $EHJG$, $DGIF$ and $GJLI$. Thus $DG||BE$ and $GJ||EH$; the fact that $B$, $E$ and $H$ are collinear implies that $D$, $G$ and $J$ are as well. A similar argument shows $F$, $I$ and $L$ are also collinear. Since $I$ is on the line segment $FL$ and also the circumcenter of $\triangle FKL$, we have $\angle FKL = 90^\circ$. Bottles $K$ and $A$ are both tangent to the rectangular rack, and so $KL||AB$. Similarly, $ML||CB$. Since the first layer $A$, $B$, $C$ is horizontal, this shows that the centers $K$, $L$, $M$ are collinear and the fifth layer of bottles is horizontal.

**H2.** You are given a safe with the lock consisting of a $4 \times 4$ arrangement of keys. Each of the 16 keys can be in a horizontal or a vertical position. To open the safe, all the keys must be in the vertical position. When you turn a key, all the keys in the same row and column change positions. You may turn a key more than once.

(a) Given the configuration in the figure below, how do you open the safe?

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(b) If you are allowed to turn at most 2002 keys, what is the largest $2n \times 2n$ safe that you can always open?

*Problem from Ross Honsberger’s private collection.*

We received two solutions out of which we present the one by Joel Schlosberg, slightly modified by the editor.

(a) First turn the key in the first row from the top and the second column from the left, then the key in the third row from the top and the third column from the left.

(b) If we choose a key and turn it, as well as each of the $4n - 2$ keys that are either in the same row or the same column as that key, then this key changes position $4n - 1$ times and thus switches its position. The keys in the same row or column change position $2n$ times and all remaining keys change position 2 times. The only key changed from its original position is the chosen key. By applying this method to each key in the horizontal position, any configuration of the safe can be unlocked.

Note that the order in which keys are turned does not affect the result. Furthermore any two turns of the same key cancel out each other’s effect, so each particular key need only be turned at most once. There are $2^{2n^2}$ possible con-
figurations for the \((2n)^2\) safe keys (since each key can be in one of two possible positions, horizontal or vertical) and \(2^{(2n)^2}\) ways to turn each of the \((2n)^2\) keys at most once. Since for each configuration there exists a way to turn the keys to unlock the safe, each configuration uniquely corresponds to one of the ways to turn the keys.

The configuration needing the most key turns (all \((2n)^2\) keys) is the one with all keys in the horizontal position. In particular, a \(46 \times 46\) safe can require \(46^2 = 2116\) > 2002 key turns to open, while a \(44 \times 44\) safe can be opened with at most \(44^2 = 1936\) < 2002 key turns and is thus the largest \(2n \times 2n\) safe that can always be opened with 2002 key turns.

**H3.** Clearly in the left figure below, the four corners of a square can be folded over to meet at a point without overlapping or gaps; another such quadrilateral is illustrated in the figure on the right.

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{figure.png}
\end{array}
\]

Determine the necessary and sufficient conditions for such a folding of the corners of a quadrilateral.

*Problem from Ross Honsberger’s private collection.*

We received four correct submissions. All solvers followed essentially the same approach presented below.

It is necessary and sufficient that the quadrilateral be convex and its diagonals are perpendicular. The convexity is clear, and we assume this property.

Suppose that the folding is possible. Let \(A, B, C, D\) be the vertices of the quadrilateral and \(P, Q, R, S\) the points where the folds meet the respective sides \(AB, BC, CD, DA\). If the folds meet at \(O\), then \(AP = OP = BP\), so that \(P\) is the midpoint of \(AB\) and triangle \(AOB\) is right. Hence \(AO \perp BO\). Similarly \(BO \perp CO\), \(CO \perp DO\) and \(DO \perp AO\). Thus \(AOC\) and \(BOD\) are perpendicular straight lines constituting the diagonals.

On the other hand, suppose that the diagonals \(AC\) and \(BD\) are perpendicular, meeting at \(O\), and that \(P, Q, R, S\) are the midpoints of the sides. Then \(PS \parallel BD\), \(PS \perp AC\) and \(PS\) is equidistant from \(A\) and \(O\). Since \(O\) is the reflected image of \(A\) about \(PS\), triangles \(APS\) and \(OPS\) are congruent. Similarly \(\Delta BQP \equiv \Delta OQP\), \(\Delta CRQ \equiv \Delta ORQ\) and \(\Delta DSR \equiv \Delta ORS\), so that triangles \(APS\), \(BQP\), \(CRQ\) and \(DSR\) fold in to exactly cover rectangle \(PQRS\).

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**H4.** Tangents to the incircle of \( \triangle ABC \) are drawn parallel to the sides to cut off a little triangle at each vertex (see figure).

Prove that the inradii of the three small circles add up to the inradius of \( \triangle ABC \); that is \( r_1 + r_2 + r_3 = r \).

From “Problems in plane and solid geometry” by Viktor Prasolov.

We received nine submissions; all of them were based on the observation that the three small triangles are similar to the given triangle \( \triangle ABC \), which was likely the “Aha moment” that Ross Honsberger found so attractive about the problem. We present a composite of the similar solutions from C.R. Pranesachar, Joel Scholsberg, Titu Zvonaru, and the Missouri State University Problem Solving Group.

Let \( h_a \) be the length of the altitude of \( \triangle ABC \) from \( A \) to \( BC \). The length of the corresponding altitude from \( A \) to the opposite side of the little triangle at \( A \) is \( h_a - 2r \), since its side opposite \( A \) is on a line parallel to \( BC \) and closer to \( A \) by a distance of \( 2r \). Note that because their corresponding sides are parallel, the two triangles are similar; exploiting that similarity, we see that

\[
\frac{r_1}{r} = \frac{h_a - 2r}{h_a}.
\]

Since the area of \( \triangle ABC \) is given by both \( \frac{1}{2}ah_a \) and \( \frac{1}{2}(a + b + c)r \),

\[
\frac{h_a - 2r}{h_a} = \frac{(a + b + c) - 2a}{a + b + c}.
\]

Similarly

\[
\frac{r_2}{r} = \frac{(a + b + c) - 2b}{a + b + c}, \quad \text{and} \quad \frac{r_3}{r} = \frac{(a + b + c) - 2c}{a + b + c}.
\]

Consequently,

\[
\frac{r_1 + r_2 + r_3}{r} = \frac{3(a + b + c) - 2a - 2b - 2c}{a + b + c} = 1,
\]

as desired.

**Editor’s comments.** Along with his solution, J. Chris Fisher observed that H4 is closely related to Problem 2.6.4 in H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems: San Gaku*, namely,

\( I(r) \) is the incircle of triangle \( ABC \). The three circles \( O_i(r_i) \) (\( i = 1, 2, 3 \)) are such that \( O_1(r_1) \) passes through \( B \) and \( C \) and touches \( I(r) \), surrounding it, and the other two behave in a similar fashion. The circles \( O'_i(r'_i) \), (\( i = 1, 2, 3 \)) are such that \( O'_1(r'_1) \) touches \( AB \) and \( AC \) and touches \( O_1(r_1) \) externally, and the other two circles behave similarly. Show that \( r'_1 + r'_2 + r'_3 = r \).
A graphics program suggests that the three small circles are the same in both problems, specifically,

For any triangle $ABC$ let $\alpha$ be the circle through $B$ and $C$ that surrounds the incircle and is tangent to it, while $\beta$ is a circle tangent to the sides $AB$ and $AC$. Then $\beta$ is externally tangent to $\alpha$ if and only if it is also tangent to the line parallel (but not equal) to $BC$ that is tangent to the incircle.

Does anybody see how to prove this conjecture?

**H5.** A circle is divided into equal arcs by $n$ diameters (see the figure). Prove that the feet of the perpendiculars to these diameters from a point $P$ inside the circle always determine a regular $n$-gon $N$.

*Problem from Ross Honsberger’s private collection.*

*We received six correct solutions from five individuals. All solvers followed essentially the same approach presented below.*

The problem can be solved for any point $P$ in the plane except the centre $O$ of the circle. Let $X$ be the foot of the perpendicular from $P$ to one of the diameters extended. Since $\angle PXO = 90^\circ$, $X$ lies on the circle with diameter $PO$. Thus, each vertex of $N$ lies on the same circle with diameter $OP$. Each edge of $N$ subtends at $O$ the common angle $2\pi/n$ between adjacent diameters of the given circle. Hence, the edges of $N$ are equal and $N$ is a regular $n$-gon.

*Editor’s comments.* The Missouri State University Problem Solving Group used complex numbers, beginning with the formula to determine the orthogonal projection $(z + \bar{w}/\bar{w})/2$ of $z$ onto the complex number $w$. With $z$ representing $P$ and the given circle centred at the origin, the vertices of $N$ are found to be

$$(z/2) + (\bar{z}/2)(\cos 2k\pi/n + i\sin 2k\pi/n), \quad (0 \leq k \leq n - 1).$$

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