OLYMPIAD SOLUTIONS


OC316. Let $ABC$ be a right-angled triangle with $\angle B = 90^\circ$. Let $BD$ be the altitude from $B$. Let $P, Q$ and $I$ be the incenters of triangles $ABD, CBD$ and $ABC$ respectively. Show that the circumcenter of $PIQ$ lies on the hypotenuse $AC$.

Originally 2015 India National Olympiad, Problem 1.

We received 7 solutions. We present the solution by Mohammed Aassila, slightly modified by the editor.

Denote by $\Gamma$ the circumcircle of $\triangle PQD$, and by $T$ the other point (besides $D$) where $\Gamma$ intersects $AC$.

Since $DP$ and $DQ$ are angle bisectors, and $BD \perp AC$, we have $\angle PDQ = 90^\circ$ and $\angle QDC = 45^\circ$. By construction, $PDTQ$ is a cyclic quadrilateral, whence $\angle PTQ = \angle PDQ = 90^\circ$ and $\angle QPT = \angle QDT = \angle QDC = 45^\circ$. Hence $\Delta PTQ$ is an isosceles right-angled triangle. Draw the circle $\Sigma$ that goes through $P$ and $Q$ and has center $T$. Note that the major arc $PQ$ has length $270^\circ$.

The points $A, P, I$ are collinear (they are all on the bisector of $\angle A$), and so are $C, Q, I$. Hence $\angle PIQ = \angle AIC = 135^\circ$, which is half of the major arc $PQ$, implying $I$ is on $\Sigma$. Therefore the circle $\Sigma$ is the circumcircle of $\triangle PIQ$, and the center of $\Sigma$ (namely, $T$) is on the hypotenuse $AC$.

Editor’s comments. As pointed out in some of the other solutions, the center $T$ of the circle $\Sigma$ is also the point of tangency of the incircle of $\triangle ABC$ to the hypotenuse $AC$.

OC317. In a recent volleyball tournament, 110 teams participated. Every team has played every other team exactly once (there are no ties in volleyball). It turns out that in any set of 55 teams, there is one which has lost to no more than 4 of the remaining 54 teams. Prove that in the entire tournament, there is a team that has lost to no more than 4 of the remaining 109 teams.

Originally 2015 All Russian Olympiad Grade 11, Problem 3.
We received two solutions, out of which we present the one by Oliver Geupel, slightly modified by the editor.

Let \( P(n) \) denote the assertion that in every subset of \( n \) teams there is one that has lost to no more than 4 of the remaining \( n - 1 \) teams. We prove \( P(n) \) for \( 55 \leq n \leq 110 \) by induction.

The base case \( n = 55 \) is ensured by the hypothesis.

Let us assume that \( P(n) \) holds for some \( n \) with \( 55 \leq n \leq 109 \) and consider a set \( S \) of \( n + 1 \) teams. We have to show that there is a team in \( S \) that has lost to no more than 4 of the remaining \( n \) teams in \( S \).

Let \( T_1, T_2, \ldots, T_m \) be all teams in \( S \) that have lost to no more than 5 teams in \( S \). If there is a \( T_i \) that has lost to no more than 4 teams in \( S \) we are done, thus we may assume that each \( T_i \) has lost to exactly 5 teams in \( S \).

Any team \( T \in S \) has won against some \( T_i \), as otherwise there could be no team in the \( n \)-set \( S \setminus \{T\} \) that lost to 4 or less teams, contradicting the induction hypothesis. Since each team \( T_i \) lost to exactly 5 teams in \( S \), we thus have \( m \geq n + 1 \). As \( n \geq 56 \) we obtain \( m \geq 12 \). But some \( T_i \) must have lost to at least \( \lceil m/2 \rceil \geq 6 \) of the remaining \( T_j \), a contradiction. As a consequence the assumption that every \( T_i \) has lost to exactly 5 of the remaining teams in \( S \) was false which completes the induction step.

**OC318.** Let \( n \) be a positive integer and let \( k \) be an integer between 1 and \( n \) inclusive. Given an \( n \times n \) white board, we do the following process.

We draw \( k \) rectangles with integer side lengths and sides parallel to the sides of the \( n \times n \) board, and such that each rectangle covers the top-right corner of the \( n \times n \) board. Then, the \( k \) rectangles are painted black. This process leaves a white figure in the board.

How many different white figures can be formed with \( k \) rectangles that cannot be formed with less than \( k \) rectangles?

*Originally 2015 Mexico National Olympiad Day 2, Problem 2.*

We received two solutions. We present the one by Oliver Geupel, expanded by the editor.

We show that there are \( \binom{n}{k}^2 \) such figures. Consider Cartesian coordinates where the origin is the lower left corner of the white board, the axes are parallel to the sides of the board and the top right corner has coordinates \((n,n)\). Let us consider \( k \) black rectangles with lower left corners \((x_1,y_1), \ldots, (x_k,y_k)\). Without loss of generality, \( x_1 \leq \cdots \leq x_k \). We claim that the figure cannot be formed with less than \( k \) rectangles if and only if

\[
x_1 < \cdots < x_k \text{ and } y_1 > \cdots > y_k.
\] (1)

If (1) holds then the figure cannot be formed with less than \( k \) rectangles as each of the \( k \) points \((x_i,y_i)\) has to be covered by a different rectangle. Suppose (1)
does not hold. Then there are rectangles $i, j$ with $x_i \leq x_j$ and $y_i \leq y_j$. Then rectangle $j$ is contained in rectangle $i$ and the same figure can be formed by using all rectangles except rectangle $j$.

As $0 \leq x_i, y_i \leq n - 1$, there are $\binom{n}{k}$ choices for $x_1, \ldots, x_k$ and $\binom{n}{k}$ choices for $y_1, \ldots, y_k$. Since the $x_i$ and $y_j$ can be chosen independently of each other, the result follows.

**OC319.** Let $p > 30$ be a prime number. Prove that one of the following numbers

$$p + 1, 2p + 1, 3p + 1, \ldots, (p - 3)p + 1$$

is the sum of two integer squares $x^2 + y^2$ for integers $x$ and $y$.

*Originally from the 2015 Iranian Mathematical Olympiad.*

*We received 2 solutions. We present the solution by Steven Chow.*

We prove the stronger result:

Let $p \geq 7$ be a prime number. Let

$$S = \left\{ np + 1 : 1 \leq j \leq \frac{p - 5}{2} \text{ is an integer} \right\}.$$

Prove that one of the numbers in $S$ is the sum of 2 square numbers, i.e., is equal to $x^2 + y^2$ for some integers $x$ and $y$.

The Legendre symbol and some well known facts are used.

Since $p \geq 7$ is prime, $p$ is congruent to either 1, 3, 5, or 7 modulo 8.

If $p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}$, then $p \geq 11$ and either $(-1/p) = 1$ and $(2/p) = 1$, or $(-1/p) = -1$ and $(2/p) = -1$, so $(-8/p) = (-1/p)(2/p) = 1$. So there exists an integer $0 \leq y \leq \frac{p - 1}{2}$ such that $3^2 + y^2 \equiv 1 \pmod{p}$, and

$$p \geq 11 \implies \left( \frac{p - 5}{2} \right) p + 1 \geq 3^2 + \left( \frac{p - 1}{2} \right)^2 \geq 3^2 + y^2,$$

so $3^2 + y^2 \in S$.

If $p = 13$, then $2^2 + 6^2 = 3p + 1 \in S$.

If $p \equiv 5 \pmod{8}$ and $p \neq 13$, then $p \geq 29$ and $(2/p) = -1$, so $(-24/p) = (2/p)(-3/p) = -(-3/p)$, so there exists an integer $0 \leq y \leq \frac{p - 1}{2}$ such that either $2^2 + y^2$ or $5^2 + y^2$ is congruent to 1 (mod $p$), and

$$p \geq 29 \implies \left( \frac{p - 5}{2} \right) p + 1 > 5^2 + \left( \frac{p - 1}{2} \right)^2 \geq 5^2 + y^2, 2^2 + y^2,$$

so either $2^2 + y^2 \in S$ or $5^2 + y^2 \in S$.

Assume now that $p \equiv 7 \pmod{8}$.
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Since \( p \equiv 7 \pmod{8} \) is prime, \((-1/p) = -1\) and \((2/p) = 1\).

Let \( q \) be the least prime such that \((q/p) = -1\), so \( q \geq 3 \) and \( p \nmid q \).

If \( q > p \) or \( q \) does not exist, then from prime-power factorization, for all integers \( 2 \leq j \leq p - 1 \), \((j/p) = 1\), which is a contradiction (since any square number is congruent modulo \( p \) to an element of \( \{j^2 : 0 \leq j \leq \frac{p-1}{2} \} \) which has cardinality \( \frac{p+1}{2} < p - 2 \) since \( p \geq 7 \)).

Therefore \( 3 \leq q < p \) and

\[
\left( \frac{-(q-1)^2+1}{p} \right) = -\left( \frac{(q-1)^2-1}{p} \right) \]

\[
= -\left( \frac{(q-2)}{p} \right) \left( \frac{q}{p} \right) \]

\[
= \left( \frac{(q-2)}{p} \right) \]

\[
= 1
\]

from the prime-power factorization of \( q - 2 \) and the definition of \( q \) (if \( q = 3 \), then \( q - 2 = 1 \)).

Therefore there exists integers \( 1 \leq x, y \leq \frac{p-1}{2} \) such that \( x^2 \equiv (q-1)^2 \pmod{p} \) and \( x^2 + y^2 \equiv 1 \pmod{p} \).

Since \( p \geq 7 \) is prime,

\[
\left( \frac{p-1}{2} \right)^2 + \left( \frac{p-1}{2} \right)^2 \equiv \frac{1}{2} \pmod{p}
\]

is not congruent to 1 \((\pmod{p})\), and

\[
\left( \frac{p-3}{2} \right)^2 + \left( \frac{p-1}{2} \right)^2 \equiv \frac{5}{2} \pmod{p}
\]

is not congruent to 1 \((\pmod{p})\), so

\[
p \geq 7 \implies \left( \frac{p-5}{2} \right) p + 1 \geq \left( \frac{p-3}{2} \right)^2 \cdot 2 \geq x^2 + y^2,
\]

so \( x^2 + y^2 \in S \).

Therefore one of the numbers in \( S \) is the sum of 2 square numbers. \( \square \)

**OC320.** Let \( n \geq 2 \) be a given integer. Initially, we write \( n \) sets on the blackboard and do a sequence of *moves* as follows:

- choose two sets \( A \) and \( B \) on the blackboard such that neither of them is a subset of the other, and replace \( A \) and \( B \) by \( A \cap B \) and \( A \cup B \).

Find the maximum number of moves in a sequence for all possible initial sets.

*Originally 2015 China Girls Mathematics Olympiad Day 2, Problem 8.*

*No solutions received.*

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