No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


4221. Proposed by Nguyen Viet Hung.

Let $a, b, c, p, q$ be distinct positive real numbers satisfying

\[
\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} = p, \\
\frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} + \frac{1}{(a-b)^2} = q.
\]

Evaluate

\[
\frac{a}{(b-c)^2} + \frac{b}{(c-a)^2} + \frac{c}{(a-b)^2}
\]

in terms of $p$ and $q$.

We received 13 submissions, 12 of which were correct, and we present the same solutions by Michel Bataille and Prithwijit De.

Let $r = \frac{a}{(b-c)^2} + \frac{b}{(c-a)^2} + \frac{c}{(a-b)^2}$. We show that $r = \sqrt{q(p-2)}$.

Let

\[
s = \frac{1}{(c-a)(a-b)} + \frac{1}{(a-b)(b-c)} + \frac{1}{(b-c)(c-a)},
\]
\[
t = \frac{a}{(c-a)(a-b)} + \frac{b}{(a-b)(b-c)} + \frac{c}{(b-c)(c-a)},
\]
\[
u = \frac{bc}{(c-a)(a-b)} + \frac{ca}{(a-b)(b-c)} + \frac{ab}{(b-c)(c-a)}.
\]

Then

\[
s = \frac{(b-c) + (c-a) + (a-b)}{(a-b)(b-c)(c-a)} = 0,
\]
\[
t = \frac{a(b-c) + b(c-a) + c(a-b)}{(a-b)(b-c)(c-a)} = 0,
\]
\[
u = \frac{bc(b-c) + ca(c-a) + ab(a-b)}{(a-b)(b-c)(c-a)} = -1.
\]

Hence,

\[
\left( \frac{1}{b-c} + \frac{1}{c-a} + \frac{1}{a-b} \right)^2 = q + 2s = q
\]

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and \[ \left( \frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} \right)^2 = p + 2u = p - 2. \]

Thus, \[
q(p - 2) = \left( \left( \frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} \right) \left( \frac{1}{b-c} + \frac{1}{c-a} + \frac{1}{a-b} \right) \right)^2
= \left( r + \frac{b+c}{(c-a)(a-b)} + \frac{c+a}{(a-b)(b-c)} + \frac{a+b}{(b-c)(c-a)} \right)^2
= (r + (a + b + c)s - t)^2
= r^2.
\]

Since \( p > 2 \) and \( r > 0 \), \( r = \sqrt{q(p - 2)} \) follows.

Editor's comment. Steven Chow pointed out that the solution to this problem includes the proof of problem 1 of the 2017 Canadian Mathematical Olympiad which asks to show that \( p > 2 \) and is, in essence, a sub-problem of the current problem.

4222. Proposed by Mihaela Berindeanu.

Let \( ABCD \) be a quadrilateral inscribed in a circle and \( X \) a mobile point on the small arc \( CD \). If \( E, F, G, H \) are the orthogonal projections of \( X \) on the lines \( AD, BC, AC, BD \) show that the angle between \( EH \) and \( GF \) is always constant, regardless of the position of \( X \) on the arc.

Soit \( ABCD \) un quadrilatère inscrit dans un cercle et soit \( X \) un point situé sur le petit arc \( CD \). Si \( E, F, G \) et \( H \) sont les projections orthogonales de \( X \) vers \( AD, BC, AC \) et \( BD \), démontrer que l’angle entre \( EH \) et \( GF \) est constant, quel que soit le point \( X \) sur l’arc.

We received 12 submissions, all correct; most of the solutions were quite similar to our featured solution by Jean-Claude Andrieux, which was singled out to remind our readers that solutions can be submitted in either of our two official languages.

Rappelons le théorème de la droite de Simson: Soit \( ABC \) un triangle quelconque et \( M \) un point du plan. On note \( P, Q \) et \( R \) les projetés orthogonaux de \( M \) respec-
tivement sur \((AB)\), \((BC)\) et \((CA)\); alors, \(P\), \(Q\) et \(R\) sont alignés si et seulement si \(M\) appartient au cercle circonscrit à \(ABC\).

Dans le problème posé, notons \(Y\) le projeté orthogonal de \(X\) sur \((AB)\).

Considérons le triangle \(ABC\): \(X\) appartient au cercle circonscrit au triangle \(ABC\) donc les points \(F\), \(G\) et \(Y\) projetés orthogonaux de \(X\) respectivement sur \((BC)\), \((CA)\) et \((AB)\) sont alignés.

Considérons le triangle \(ABD\): \(X\) appartient au cercle circonscrit au triangle \(ABD\) donc les points \(E\), \(H\) et \(Y\) projetés orthogonaux de \(X\) respectivement sur \((AD)\), \((BD)\) et \((AB)\) sont alignés.

Les droites \((FG)\) et \((EH)\) se coupent donc en \(Y\). Il faut montrer que l’angle \(\widehat{EYF}\) est indépendant de la position de \(X\) sur l’arc \(CD\).

Les triangles \(XEA\), \(XGA\) et \(XYA\) sont des triangles rectangles de même hypoténuse \([XA]\). Les points \(X, E, A, G\) et \(Y\) sont donc cocycliques.

On a alors: \(\widehat{EYF} = \widehat{EYG} = \widehat{EAG} = \widehat{DAC}\).

L’angle \(\widehat{EYF}\) est donc constant et on a \(\widehat{EYF} = \widehat{DAC} = \widehat{DBC}\).

**Editor’s comments.** Steven Chow observed that if directed angles are used (modulo \(\pi\), the point \(X\) need not be restricted to the small arc \(CD\): the angle between \(EH\) and \(GF\) remains constant (and the featured proof remains valid) for all positions of \(X\) on the circle. Somasundaram Muralidharan observed, similarly, that the final line of our argument shows that while \(C\) and \(D\) must be fixed points, \(A\) and \(B\) are free to move about the circle without changing the angle between \(EH\) and \(GF\). Chow also suggested that the editors perhaps should not have included the proposer’s diagram with the statement of his problem since it essentially provides the solution. Maybe that explains the similarity of so many of the submissions. Bataille’s solution, however, was based on the spiral similarity with fixed point \(X\) that takes \(C\) to \(D\) (and therefore \(G\) to \(E\) and \(F\) to \(H\)). He added to his solution the observation that the lines \(EG, FH\), and \(CD\) are concurrent in a point common to the circles on diameter \(XC\) and \(XD\). For more information about arguments that exploit intersecting circles and spiral similarities, see his article “Focus On... No. 12” [40:5 (May 2014) 203-206].

**4223.** Proposed by Leonard Giugiuc and Dorin Marghidanu.

Let \(a, b\) and \(c\) be positive real numbers such that \(a + b + c \leq 1\). Prove that

\[
\sqrt[3]{(1 - a^3)(1 - b^3)(1 - c^3)} \geq 26abc.
\]

We received 15 correct solutions and we present a very succinct proof by Titu Zeonaru.

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Since \(1 \geq (a + b + c)^3\), we have by the AM-GM inequality that
\[
1 - a^3 \geq (a + b + c)^3 - a^3 = b^3 + c^3 + 3(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2) + 6abc \geq 26 \sqrt[3]{a^{32}b^8c^8},
\]
with equality if and only if \(a + b + c = 1\) and \(a = b = c\) or two of \(a, b,\) and \(c\) are 0.

Multiplying by the other two inequalities obtained by considering \(1 - b^3\) and \(1 - c^3\) we then obtain
\[
(1 - a^3)(1 - b^3)(1 - c^3) \geq (26)^3 \sqrt[3]{a^{32}b^8c^8},
\]
from which \(\sqrt[3]{(1 - a^3)(1 - b^3)(1 - c^3)} \geq 26abc\) follows.

The equality holds if and only if \((a, b, c) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (1, 0, 0), (0, 1, 0),\) or \((0, 0, 1)\).

**Editor’s comments.** Geupel remarked that the given inequality can be generalized to the following result:

if \(n\) is a natural number and \(a_1, a_2, \ldots, a_n\) are positive real numbers such that \(a_1 + a_2 + \cdots + a_n \leq 1\), then
\[
\sqrt[3]{(1 - a_1^n)(1 - a_2^n) \cdots (1 - a_n^n)} \geq (n - 1)a_1a_2 \cdots a_n.
\]

4224. **Proposed by Michel Bataille.**

Find the complex roots of the polynomial
\[
16x^6 - 24x^5 + 12x^4 + 8x^3 - 12x^2 + 6x - 1.
\]

We received 16 solutions. We present 2 solutions.

**Solution 1, by Prithwijit De.**

Observe that
\[
16x^6 - 24x^5 + 12x^4 + 8x^3 - 12x^2 + 6x - 1 = (2x^2)^3 + (2x^2)^3 + (2x - 1)^3 - 3(2x^2)(2x^2)(2x - 1).
\]

Using the identity
\[
a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca),
\]
we get
\[
(2x^2)^3 + (2x^2)^3 + (2x - 1)^3 - 3(2x^2)(2x^2)(2x - 1) = (4x^2 + 2x - 1)(2x^2 - 2x + 1)^2.
\]

Thus the roots (ignoring multiplicity) are \(x = \frac{-1 \pm \sqrt{5}}{4}, \pm i\).

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Solution 2, by Somasundaram Muralidharan.

Let \( f(x) = 16x^6 - 24x^5 + 12x^4 + 8x^3 - 12x^2 + 6x - 1 \). Let us first check whether \( f \) has repeated roots. Such repeated roots, if any, will be roots of \( \gcd(f(x), f'(x)) \), where \( f'(x) \) is the derivative of \( f(x) \). In this case, it is easy to see that \( \gcd(f(x), f'(x)) = 2x^2 - 2x + 1 \) and hence the roots of this \( \gcd \), namely \( \frac{1 \pm i}{2} \), are double roots of \( f(x) = 0 \). Thus we have found four of the roots of \( f \). We now find the remaining two roots of \( f \). We have

\[
  f(x) = (2x^2 - 2x + 1)^2(4x^2 + 2x - 1)
\]

and hence the remaining roots are roots of \( 4x^2 + 2x - 1 = 0 \). These are \( -\frac{1 \pm \sqrt{5}}{4} \).

So, the complex roots of \( f \) are

\[
\begin{align*}
  &-\frac{1 + \sqrt{5}}{4}, & -\frac{1 - \sqrt{5}}{4}, & \frac{1 + i}{2}, & \frac{1 - i}{2}, & \frac{1 - i}{2}.
\end{align*}
\]


Prove that in any triangle \( ABC \) we have:

\[
  3(\cos^2 A + \cos^2 B + \cos^2 C) + \cos A \cos B + \cos A \cos C + \cos B \cos C \geq 3.
\]

We received 13 correct solutions. We present the solution by Arkady Alt.

Since \( \cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1 \) in any triangle \( ABC \), the original inequality is successively equivalent to

\[
  3(1 - 2 \cos A \cos B \cos C) + \cos A \cos B + \cos A \cos C + \cos B \cos C \geq 3
\]

\[
\iff \cos A \cos B + \cos A \cos C + \cos B \cos C \geq 6 \cos A \cos B \cos C.
\]

Setting \( t = \sqrt[3]{\cos A \cos B \cos C} \) and using the AM-GM inequality, we obtain

\[
1 - 2t^3 = 1 - 2 \cos A \cos B \cos C
\]

\[
= \cos^2 A + \cos^2 B + \cos^2 C
\]

\[
\geq 3 \cdot \sqrt[3]{\cos^2 A \cdot \cos^2 B \cdot \cos^2 C}
\]

\[
= 3t^2.
\]

Therefore \( 2t^3 + 3t^2 - 1 \leq 0 \), implying successively that \( (2t - 1)(t + 1)^2 \leq 0 \) and then \( t \leq \frac{1}{2} \). Hence, \( 2 \cdot \sqrt[3]{\cos A \cos B \cos C} \leq 1 \), and again by the AM-GM inequality we have

\[
\cos A \cos B + \cos A \cos C + \cos B \cos C \geq 3 \cdot \sqrt[3]{\cos^2 A \cdot \cos^2 B \cdot \cos^2 C}
\]

\[
\geq 3 \cdot \sqrt[3]{\cos^2 A \cdot \cos^2 B \cdot \cos^2 C} \cdot \sqrt[3]{\cos A \cos B \cos C}
\]

\[
= 6 \cos A \cos B \cos C.
\]
4226. Proposed by Daniel Sitaru.

Prove that if $0 < a < b$ then:

$$
\left( \int_a^b \frac{\sqrt{1 + x^2}}{x} \, dx \right)^2 > (b - a)^2 + \ln^2 \left( \frac{b}{a} \right).
$$

We received nine submissions, eight of which are correct and the other is incorrect. We present a composite of virtually the same solutions by Arkady Alt; Michel Bataille; M. Bello, M. Benito, O. Ciaurri, E. Fernández, and L. Roncal (jointly); and Digby Smith.

Note first that

$$
\int_a^b \frac{\sqrt{1 + x^2}}{x} \, dx > (b - a)^2 + \ln \left( \frac{b}{a} \right).
$$

$$
\iff \int_a^b \frac{\sqrt{1 + x^2}}{x} \, dx - \int_a^b \frac{1}{x} \, dx > (b - a)^2
$$

$$
\iff \int_a^b \frac{\sqrt{1 + x^2} + 1}{x} \, dx \cdot \int_a^b \frac{\sqrt{1 + x^2} - 1}{x} \, dx > (b - a)^2. \quad (1)
$$

Let $f(x) = \frac{\sqrt{1 + x^2} + 1}{x}$, $x \in [a, b]$. Then $f(x) > 0$ and $\frac{1}{f(x)} = \frac{\sqrt{1 + x^2} - 1}{x}$. By the integral form of the Cauchy-Schwarz Inequality, we have

$$
\left( \int_a^b f(x) \, dx \right) \left( \int_a^b \frac{1}{f(x)} \, dx \right) = \left( \int_a^b \left( \sqrt{f(x)} \right)^2 \, dx \right) \left( \int_a^b \left( \frac{1}{f(x)} \right)^2 \, dx \right)
$$

$$
\geq \left( \int_a^b 1 \, dx \right)^2
$$

$$
= (b - a)^2. \quad (2)
$$

But equality cannot hold in (2) as $f$ is not a constant on $[a, b]$. Hence, from (1) and (2) the result follows.

4227. Proposed by Dan Marinescu and Leonard Giugiuc.

Let $P$ be a point in the interior of an equilateral triangle $ABC$ whose sides have length 1, and let $R'$ and $r'$ be the circumradius and inradius of the triangle whose sides are congruent to $PA$, $PB$ and $PC$ (which exists by Pompeiu’s theorem). Prove that

$$
3R' \geq 1 \geq 6r'.
$$

Among the four submissions, three were complete and correct; in the fourth, Michel Bataille simply provided a reference where the proof can be found: Proposition 7 in Józef Sándor’s “On the Geometry of Equilateral Triangles”, Forum Geometricorum, vol. 5 (2005) 107-117. Here we present the solution by Roy Barbara.
Let square brackets denote area and let \( T = \Delta A'B'C' \) denote the given Pompeiu triangle with sides \( a = PA, b = PB, c = PC \) (with \( a \) opposite \( A' \), etc.).

**Lemma.**

\[
r' = \frac{\sqrt{3}}{2} \left( \frac{2 - (a^2 + b^2 + c^2)}{a + b + c} \right).
\]

**Proof of Lemma.** Let \( X, Y, \) and \( Z \) be the reflections of \( P \) through \( BC, CA, \) and \( AB \), respectively. Triangle \( AZY \) satisfies \( AY = AZ = a \) and \( \angle YAZ = 120^\circ \); hence,

\[
[AZY] = \frac{\sqrt{3}}{4} a^2 \quad \text{and, similarly, } [BXZ] = \frac{\sqrt{3}}{4} b^2 \quad \text{and} \quad [CYX] = \frac{\sqrt{3}}{4} c^2. \quad (1)
\]

Triangle \( XYZ \), having sides \( a\sqrt{3}, b\sqrt{3}, c\sqrt{3} \) is similar to \( T \). Clearly, the area of the hexagon \( AZBXCY \) is twice \([ABC]\) so,

\[
[AZBXCY] = \frac{\sqrt{3}}{2}. \quad (2)
\]

But also, \([AZBXCY] = [XYZ] + [AZY] + [BXZ] + [CYX]. \) From this, (1), and (2) we get

\[
[XYZ] = \frac{\sqrt{3}}{4} \left( 2 - (a^2 + b^2 + c^2) \right). \quad (3)
\]

The inradius \( r'' \) of \( \Delta XYZ \) is \( \frac{[XYZ]}{s''} \), where \( s'' = \frac{\sqrt{3}}{2} (a + b + c) \) is the semiperimeter of \( \Delta XYZ \). From this and (3) we get

\[
r'' = \frac{1}{2} \left( \frac{2 - (a^2 + b^2 + c^2)}{a + b + c} \right). \]

Finally, since \( T \) is similar to \( \Delta XYZ \) with ratio \( \frac{1}{\sqrt{3}} \), we obtain \( r' = \frac{1}{\sqrt{3}} r'' \), and the lemma follows.

**Proof that** \( R' \geq \frac{1}{3} \). Since \( \Delta ABC \) is equilateral, its Fermat point (that minimizes \( a + b + c \)) is its centroid. Hence,

\[
a + b + c \geq \sqrt{3}. \quad (4)
\]

By the Law of Sines we have

\[
2R' = \frac{a}{\sin A'} = \frac{b}{\sin B'} = \frac{c}{\sin C'} = \frac{a + b + c}{\sin A' + \sin B' + \sin C'}. \quad (5)
\]

Further, any triangle \( A'B'C' \) satisfies \( \sin A' + \sin B' + \sin C' \leq \frac{3\sqrt{3}}{2} \). This together with (5) and (4) yields

\[
2R' = \frac{a + b + c}{\sin A' + \sin B' + \sin C'} \geq \frac{\sqrt{3}}{\left( \frac{3\sqrt{3}}{2} \right)} = \frac{2}{3};
\]

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that is, \( R' \geq \frac{1}{3} \), as desired. Equality holds if and only if \( P \) is the centroid of \( \Delta ABC \) (in which case \( \Delta A'B'C' \) is also equilateral).

**Proof that** \( r' \leq \frac{1}{6} \). Set \( S = a + b + c \). By the lemma, \( r' = \frac{\sqrt{3}}{6} \left( \frac{2 - (a^2 + b^2 + c^2)}{a + b + c} \right) \); moreover, \( a^2 + b^2 + c^2 \geq \frac{S^2}{3} \) is a basic inequality. Consequently,
\[
r' \leq \frac{\sqrt{3}}{18} \left( \frac{6 - S^2}{S} \right).
\]
(6)

By (4), \( S \geq \sqrt{3} \). Furthermore, \( \frac{6 - S^2}{S} \) is decreasing with respect to \( S \), whence
\[
\frac{6 - S^2}{S} \leq \frac{6 - (\sqrt{3})^2}{\sqrt{3}} = \sqrt{3}.
\]

The desired result follows immediately from this together with (6); also here, equality holds if and only if \( \Delta A'B'C' \) is equilateral.

**4228. Proposed by Mihály Bencze.**

Let \( z_k \in \mathbb{C}, k = 1, 2, \ldots, n \) such that \( \sum_{k=1}^{n} z_k = \sum_{k=1}^{n} z_k^2 = 0 \). Prove that
\[
n \sum_{k=1}^{n} |z_k|^2 \leq (n - 2) \left( \sum_{k=1}^{n} |z_k| \right)^2.
\]

We received three correct solutions and one incorrect solution. We present two solutions here.

**Solution 1, by the proposer.**

For each \( k \) with \( 1 \leq k \leq n \), we have that
\[
2z_k^2 = z_k^2 + (z_1 + z_2 + \cdots + z_k + \cdots + z_n)^2
\]
\[
= \sum_{k=1}^{n} z_k^2 + 2 \sum \{ z_i z_j : 1 \leq i < j \leq n; i, j \neq k \}
\]
\[
= 2 \sum \{ z_i z_j : 1 \leq i < j \leq n; i, j \neq k \}
\]
from which
\[
2|z_k|^2 \leq 2 \sum \{|z_i z_j| : 1 \leq i < j \leq n; i, j \neq k \}.
\]

Adding all these \( n \) inequalities leads to
\[
2 \sum_{k=1}^{n} |z_k|^2 \leq 2(n - 2) \sum_{1 \leq i < j \leq n} |z_i z_j|.
\]
Adding \((n - 2) \sum_{k=1}^{n} |z_k|^2\) to each side yields

\[
n \sum_{k=1}^{n} |z_k|^2 \leq (n - 2) \left( \sum_{k=1}^{n} |z_k|^2 + 2 \sum_{1 \leq i < j \leq n} |z_i z_j| \right) = (n - 2) \left( \sum_{k=1}^{n} |k| \right)^2.
\]
Solution 2, by Michel Bataille.

When \( n = 1, 2, 3 \), we have equality on both sides. For \( n = 1 \), \( z_1 = 0 \). For \( n = 2 \), \( z_1 + z_2 = z_1z_2 = 0 \) so that \( z_1 = z_2 = 0 \). For \( n = 3 \) and \( i \neq j \),

\[
2(z_i^3 - z_j^3) = (z_i - z_j)(z_i^2 + z_j^2 + (z_i + z_j)^2) = 0,
\]

so that \( z_1^2 = z_2^2 = z_3^2 \), whence \( |z_1| = |z_2| = |z_3| \).

Let \( n \geq 4 \). Then

\[
0 = |z_1 + \cdots + z_n|^2 = (z_1 + \cdots + z_n)(\overline{z_1} + \cdots + \overline{z_n}) = \sum_{k=1}^{n} |z_k|^2 + 2 \sum_{1 \leq i < j \leq n} \Re(z_i \overline{z_j}).
\]

Therefore

\[
\sum_{k=1}^{n} |z_k|^2 = -2 \sum_{1 \leq i < j \leq n} \Re(z_i \overline{z_j}) \leq 2 \sum_{1 \leq i < j \leq n} |\Re(z_i \overline{z_j})| \\
\leq 2 \sum_{1 \leq i < j \leq n} |z_i \overline{z_j}| = 2 \sum_{1 \leq i < j \leq n} |z_i||z_j| \\
= \left( \sum_{k=1}^{n} |z_k| \right)^2 - \sum_{k=1}^{n} |z_k|^2.
\]

Rearranging like terms and multiplying by \( n/2 \) yields the inequality

\[
n \sum_{k=1}^{n} |z_k|^2 \leq \frac{n}{2} \left( \sum_{k=1}^{n} |z_k| \right)^2,
\]

which is the required inequality when \( n = 4 \) and is stronger for larger \( n \).


Let \( n \) be an integer with \( n \geq 2 \) and let \( p \) be a prime number with \( p > n \). Consider an \( n \times n \) matrix \( X \) over \( \mathbb{Z}_p \) with \( X^p = I_n \). Prove that \((X - I_n)^n = O_n\).

There were 5 correct solutions. We present the solution obtained independently by Roy Barbara and Trey Smith.

Since \((X - I_n)^p = X^p - I_n^p = O_n\), \( X - I_n \) is nilpotent. But this implies that \((X - I_n)^n = O_n\).

Editor’s comment. For the linear algebra result invoked, let \( D \) be nilpotent and \( m \) the minimum exponent for which \( D^m = O_n \). If \( m \leq n \), then \( D^n = O_n \). If \( m > n \), suppose, if possible, (by Cayley’s theorem) that \( O_n = D^n + c_{n-1}D^{n-1} + \cdots + c_0D^0 \) with \( c_k \neq 0 \) for some \( 0 \leq k \leq n - 1 \). Multiply the equation by \( D^{n-k-1} \) to get a contradiction.

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Let $\triangle ABC$ be a triangle in which $\angle B = 2\angle C$ and let $M$ be the midpoint of $BC$. The internal bisector of $\angle ACB$ intersects $AM$ in $D$. Prove that $\angle CDM \leq 45^\circ$ and find $\angle C$ for which the equality holds.

Miguel Amengual Covas observed that this problem appeared as problem 1562 of Crux [1990:204], posed by Toshio Seimiya and that three solutions were given in [1991 : 252-254]. He observes further that it appears in The Olympiad Corner No. 161 [1995 : 9-10], with solution by Covas, distinct from those published in 1991 given in [1996 : 265-267].

References to further properties of triangles whose angles satisfy $\angle B = 2\angle C$ can be found in J. Chris Fisher’s “Recurring Crux Configurations 7”, [2012 : 238-240].

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**Bouncing Balls**

A pair of balls $A$ and $B$ of negligible radius (so you can treat them as points) lie on a perfectly flat surface with ball $B$ lying between ball $A$ and a wall. Ball $A$ has mass $100^n$ and ball $B$ has mass 1. Ball $A$ is pushed towards ball $B$ and, as the balls interact, we count the number of collisions.

When $n = 0$, ball $A$ strikes ball $B$ and stops. Ball $B$ bounces off the wall and returns to strike ball $A$. Ball $B$ then stops and ball $A$ rolls away into the distance. A total of 3 collisions occurred.

When $n = 1$, there are a total of 31 collisions.

When $n = 5$, there are a total of 314159 collisions.

See a pattern? Can you prove it?

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