Introduction

This number is about functional equations, especially the most famous of them, Cauchy’s equation

\[ f(x + y) = f(x) + f(y), \]

where the unknown function \( f : S \to S \) has to satisfy this equation for all elements \( x, y \) of \( S \). Here, \( S \) can be any set in which an addition has been defined. A lot of solutions to functional equations proposed in problems eventually end up with an application of known results about this equation, so it seems of interest to review these results. As usual, a choice of past problems will illustrate the various properties under consideration.

Elementary results

Let \( f \) denote a solution to equation (1) and let \( x \) be an element of \( S \). Taking \( y = x \) in (1) immediately shows that \( f(2x) = 2f(x) \) and an easy induction shows that

\[ f(nx) = nf(x) \]  

(2)

for all \( n \in \mathbb{N} \). Simple as it may appear, this result can be applied to problems. An example is provided by Crux problem 4040 [2015 : 170, 172 ; 2016 : 189]:

Find all functions \( f : \mathbb{N} \to \mathbb{N} \) such that

\[ (f(a) + b)f(a + f(b)) = (a + f(b))^2 \quad \forall a, b \in \mathbb{N}. \]

Three solutions of this interesting problem were featured, and we propose a fourth one. If \( f \) is a solution, we first observe that \( f(b + f(b)) = b + f(b) \) for any \( b \in \mathbb{N} \) \( (a = b \) in the equation). Then, substituting \( b + f(b) \) for \( b \) in the equation gives

\[ (f(a) + b + f(b))f(a + b + f(b)) = (a + b + f(b))^2 \]

while substituting \( a + b \) for \( a \) yields

\[ (f(a + b) + b)f(a + b + f(b)) = (a + b + f(b))^2. \]

The comparison leads to \( f(a + b) = f(a) + f(b) \), which takes us back to equation (1). With \( x = 1 \) in (2), we see that \( f(n) = nw \) for all \( n \in \mathbb{N} \) where we set \( f(1) = w \). To conclude, we remark that from the initial equation with \( a = b = 1 \) we obtain \( f(1 + w) = 1 + w \), that is, \( w(1 + w) = 1 + w \). Thus \( w = 1 \) and so \( f(n) = n \) for all
\( n \in \mathbb{N} \). Conversely, the identity function from \( \mathbb{N} \) to itself is obviously a solution and so is the unique solution.

Now, suppose that \( S \) is an additive group with 0 as the neutral element. Then, taking \( y = 0 \) in (1), we obtain \( f(0) = 0 \). With \( y = -x \), we obtain \( f(-x) = -f(x) \) and it follows that (2) now holds for any \( n \in \mathbb{Z} \). A good application is the following problem proposed at the 20th Austrian-Polish Mathematical Competition in 1997:

Prove that there does not exist a function \( f : \mathbb{Z} \to \mathbb{Z} \) such that

\[
    f(x + f(y)) = f(x) - y
\]

for all integers \( x \) and \( y \).

Assume that such a function \( f \) exists. Then for all integers \( x, y \), we have

\[
    f(f(x)) = f(y + f(x + f(y))) = f(y) - (x + f(y))
\]

hence \( f(f(x)) = -x \). We deduce \( f(-x) = f(f(x)) = -f(x) \) and so

\[
    f(y + x) = f(y + f(-x)) = f(y) + f(x).
\]

Again with \( w = f(1) \), we obtain \( f(x) = wx \) for any integer \( x \). However, substituting in the given equation leads to \(-w^2 y = y \) for all \( y \), which is clearly impossible.

It is worth noting that, in spite of this negative conclusion, functions from \( \mathbb{R} \) to \( \mathbb{R} \) satisfying the same equation do exist. Indeed, the consequences \( f(x + y) = f(x) + f(y) \) and \( f(f(y)) = -y \) continue to hold when \( x, y \) are any real numbers. An appeal to the axiom of choice then allows one to construct a suitable function. We refer the interested reader to the solution of problem 1690 in Mathematics Magazine, Vol. 77, No 1, February 2005.

Suppose that \( S \) is a vector space over \( \mathbb{Q} \) and let \( p \in \mathbb{Z}, q \in \mathbb{N}, x \in S \). From (2), we get

\[
    pf(x) = f(px) = f\left(\left(q \cdot \frac{p}{q}\right)x\right) = qf\left(\frac{p}{q}x\right)
\]

so that (2) is still valid if \( n \) is a rational number. As a result, the solutions to (1) in the case \( S = \mathbb{Q} \) are the functions \( x \mapsto wx \) where \( w \) is a rational number \((w = f(1))\). This is used in solving the following equation set at the 15th Irish Olympiad:

Determine all functions \( f : \mathbb{Q} \to \mathbb{Q} \) such that \( f(x + f(y)) = y + f(x) \)

for all \( x, y \in \mathbb{Q} \).

Despite the very slight change with the previous equation, this equation does have solutions, for example, the functions \( x \mapsto x \) and \( x \mapsto -x \), as is readily checked. We show that there are no other solutions. Let \( f \) be an arbitrary solution and \( a = f(0) \). The equation with \( x = y = 0 \) gives \( f(a) = a \) and then, with \( x = 0, y = a \), we obtain \( a = 2a \). Thus \( a = f(0) = 0 \) and it immediately follows that \( f(f(y)) = y \) for any rational number \( y \). Replacing \( y \) by \( f(y) \) in the equation finally shows that
(1) is satisfied for all \(x, y \in \mathbb{Q}\). We first deduce that for some \(w \in \mathbb{Q}\), \(f(x) = wx\) and then, inserting in the equation, that \(w = 1\) or \(w = -1\), completing the proof.

**The equation for** \(f : \mathbb{R} \to \mathbb{R}\)

In view of the results obtained in the previous paragraph, we expect that, in the case \(S = \mathbb{R}\), the solutions are the linear functions \(x \mapsto wx\) where \(w\) is a real number. Clearly, these functions are solutions but the converse does not hold (again, the latter can be proved with the help of the axiom of choice). However, as we shall see, assuming some additional property of \(f\), the solutions are the expected ones. Note that, \(\mathbb{R}\) being a \(\mathbb{Q}\)-vector space, any solution \(f : \mathbb{R} \to \mathbb{R}\) of (1) satisfies \(f(rx) = rf(x)\) when \(r \in \mathbb{Q}\) and \(x \in \mathbb{R}\) (in particular \(f(r) = rf(1)\)).

Suppose now that a solution \(f\) is monotone, say nondecreasing, and let \(x \in \mathbb{R}\). There exist sequences of rational numbers \((r_n)\) and \((s_n)\) such that \(r_n \leq x \leq s_n\) for all \(n \in \mathbb{N}\) and \(\lim_{n \to \infty} r_n = \lim_{n \to \infty} s_n = x\). Then \(r_nf(1) \leq f(x) \leq s_nf(1)\) for all \(n\) and taking the limits as \(n \to \infty\), we deduce \(xf(1) \leq f(x) \leq xf(1)\) and so \(f(x) = xf(1)\). Thus, the monotone solutions of Cauchy’s equation with \(S = \mathbb{R}\) are the linear functions.

Similarly, using an appropriate sequence of rational numbers, it is easy to obtain the same conclusion when monotone is replaced by continuous. Note also that a solution is continuous everywhere as soon as it is continuous at some point \(x_0\) (since \(f(x+h) - f(x) = f(x_0 + h) - f(x_0)\)).

An interesting application is the determination of the morphisms of the field \(\mathbb{R}\), that is, the mappings \(f : \mathbb{R} \to \mathbb{R}\) such that the conditions \(f(x+y) = f(x) + f(y)\), \(f(xy) = f(x)f(y)\) for all \(x, y \in \mathbb{R}\) and \(f(1) = 1\) are verified. Clearly, the identity function \(x \mapsto x\) is such a mapping. Let us show that it is the only one. We observe that if \(x > 0\), then \(f(x) = f(\sqrt{x}\sqrt{x}) = (f(\sqrt{x}))^2 \geq 0\) and deduce that \(f(v) - f(u) = f(v-u) \geq 0\) if \(v > u\). Thus \(f\) is nondecreasing on \(\mathbb{R}\) and the conclusion follows.

A call to this property can prove useful, as exemplified by a problem posed to select the Indian IMO team in 2003 [2006 : 278 ; 2007 : 284]. Here, we propose a shortened version:

Find all functions \(f : \mathbb{R} \to \mathbb{R}\) such that \(f(1) = 1\) and

\[
f(x+y) + f(x)f(y) = f(x) + f(y) + f(xy)
\]

for all \(x, y \in \mathbb{R}\).

A neat mix of the morphism conditions above!

Let \(f\) be any solution and let \((E)\) denote the equation. Taking \(x = y = 0\) in \((E)\) yields \(f(0) = 2\) or \(f(0) = 0\). However, \(f(0) = 2\) cannot hold, a contradiction with \(f(1) = 1\) being reached if we take \(x = 1, y = 0\) in \((E)\). Thus, \(f(0) = 0\).

With \(x = 1, y = -1\), the equation gives \(f(-1) = -1\) and with arbitrary \(x\) and \(y = 1\), we obtain \(f(x+1) = 1 + f(x)\). Substituting \(x-1\) for \(x\) yields \(f(x-1) = f(x)-1\).
Keeping $x$ arbitrary and taking $y = -1$ leads to $f(x - 1) = 2f(x) - 1 + f(-x)$ and comparing with the previous relation, we deduce that $f(-x) = -f(x)$, that is, $f$ is odd. Changing $y$ into $-y$ in $(E)$ and adding to $(E)$ itself shows that $f(x + y) + f(x - y) = 2f(x)$ for all $x, y \in \mathbb{R}$. It follows that $f(2x) = 2f(x)$ (with $y = x$) and then, changing $x$ into $\frac{u + v}{2}$ and $y$ into $\frac{u - v}{2}$, that $f(u + v) = f(u) + f(v)$ for all $u, v$. Returning to the equation, we deduce that $f(xy) = f(x)f(y)$ for all $x, y$. We can now conclude that $f$ is a morphism of the field $\mathbb{R}$, hence is the identity function. Conversely, the identity function is obviously a solution, hence is the unique solution.

Exercises

   Let $f$ be a function from $\mathbb{R}$ to $\mathbb{R}$. Prove that
   \[ f(x + y + f(y)) = f(x) + 2f(y) \]
   for all real $x$ and $y$ if and only if
   \[ f(x + y) = f(x) + f(y) \quad \text{and} \quad f(f(x)) = f(x) \]
   for all real $x$ and $y$.

2. (Problem 10854 of the American Mathematical Monthly, February 2001)
   Find every function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at 0 and satisfies
   \[ f(x + 2f(y)) = f(x) + y + f(y) \]
   for all real numbers $x$ and $y$. 