CONTEST CORNER
SOLUTIONS


CC261. Peter is walking through a train tunnel when he hears a train approaching. He knows that on this section of track trains travel at 60 mph. The tunnel has equally spaced marker posts, with post 0 at one end and post 12 at the other end. Peter is by post 7 when he hears the train. He quickly works out that whether he runs to the nearer end or the further end of the tunnel as fast as he can (at constant speed) he will just exit the tunnel before the train reaches him. How fast can Peter run?

*Originally Problem 1 of the 2016–2017 Scottish Mathematical Council Mathematical Challenge, Middle division.*

We received 4 submissions of which three were correct. We present the solution by Digby Smith, modified by the editor.

We let

\[ v = \text{constant running speed of Peter in miles per hour}, \]
\[ x = \text{starting distance between the train and post 12 in miles}, \]
\[ y = \text{distance between consecutive posts in miles}, \]
\[ t_{12} = \text{time taken for Peter and the train to reach post 12}, \]
\[ t_0 = \text{time taken for Peter and the train to reach post 0}. \]

Recall that, given a constant speed,

\[ \text{time} = \frac{\text{distance}}{\text{speed}}. \tag{1} \]

Peter and the train will travel a distance of 5\(y\) and \(x\), respectively, to reach post 12 in \(t_{12}\) units of time. Therefore, given (1) we have that

\[ t_{12} = \frac{5y}{v} = \frac{x}{60}. \tag{2} \]

Peter and the train will travel a distance of 7\(y\) and \(x + 12y\), respectively, to reach post 0 in \(t_0\) units of time. Therefore, given (1) we have that

\[ t_0 = \frac{7y}{v} = \frac{x + 12y}{60}. \]

Note that

\[ \frac{7y}{v} = \frac{x + 12y}{60} \implies \frac{7y}{v} - \frac{y}{5} = \frac{x}{60}. \]

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By substituting the above into (2) we have that
\[
\frac{7y}{v} - \frac{y}{5} = \frac{5y}{v},
\]
which simplifies to
\[
\frac{2}{v} = \frac{1}{5} \implies v = 10.
\]
Therefore, the fastest running speed of Peter is 10 miles per hour.

**CC262.** In the diagram, the square has two of its vertices on the circle of radius 1 unit and the other two vertices lie on a tangent to the circle. Find the area of the square.

![Diagram of a square with two vertices on a circle and two on a tangent]

*Originally Problem 4 of the 2016–2017 Scottish Mathematical Council Mathematical Challenge, Middle division.*

*We received 13 solutions, all of which were correct. We present three different approaches to this problem.*

**Solution 1, by Andrea Fanchini, modified by the editor.**

Let \( l \) be the side length of the square and let \( C \) be the center of the circle. We construct the following right angle triangle with vertices \( A, B, \) and \( C \). By construction, \( AC \) has length 1, \( CB \) has length \( l - 1 \), and \( BA \) has length \( \frac{l}{2} \).

![Diagram of a right angle triangle with vertices A, B, and C]

By the Pythagorean Theorem, we have that
\[
(l - 1)^2 + \left(\frac{1}{2}\right)^2 = 1 \implies l = \frac{8}{5}.
\]

Therefore the area of the square is \( l^2 = \frac{64}{25} \).
Solution 2, by Ángel Plaza, modified by the editor.

Let $x$ be the side length of the square and let $C$ be center of the circle. We construct the following two triangles and note the relationship between the angles subtending the same arc:

By construction, we have

\[
\sin(\alpha) = \frac{x/2}{\sqrt{(x/2)^2 + x^2}} = \frac{1}{\sqrt{5}},
\]

\[
\cos(\alpha) = \frac{x}{\sqrt{(x/2)^2 + x^2}} = \frac{2}{\sqrt{5}},
\]

\[
\sin(2\alpha) = \frac{x}{2}.
\]

Since $\sin(2\alpha) = 2 \cdot \sin(\alpha) \cdot \cos(\alpha)$ we have that $\frac{x}{2} = 2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}}$, so $x = \frac{8}{5}$. Therefore the area of the square is $x^2 = \frac{64}{25}$.

Solution 3, by Joel Schlosberg, modified by the editor.

Draw the diameter at the midpoint of tangency, and segments connecting the diameter’s endpoints to one of the vertices on the circle. By Thales’ theorem, this produces a right triangle.
It is split into two smaller triangles by one of the square’s sides; since the side is perpendicular to the diameter, the smaller triangles are also right triangles. By construction these smaller triangles are similar.

The right triangle inside the square has one leg twice the length of the other, and so the right triangle outside the square does also. Let \( s \) be the side length of the square. The triangle outside the square has a side length of \( s/2 \). This same length is twice the length of the the other leg which has length \( 2 - s \). Therefore

\[
\frac{s}{2} = 2(2 - s).
\]

Thus \( s = 8/5 \) and area of the square is \( s^2 = 64/25 \).

**CC263.** An old fashioned tram starts from the station with a certain number of men and women on board. At the first stop, a third of the women get out and their places are taken by men. At the next stop, a third of the men get out and their places are taken by women. There are now two more women than men and as many men as there originally were women. How many men and women were there on board at the start?

*Originally Problem 2 of the 2016–2017 Scottish Mathematical Council Mathematical Challenge, Middle division.*

We received six solutions to this problem. We present the composite solution of Dan Daniel, Digby Smith, and Titu Zvonaru.

Let \( m_i \) and \( w_i \) be the number of men and women on board the train, respectively, at stop \( i \). Note that \( m_0 \) and \( w_0 \) is the number of men and women on board the train, respectively, at the start of the trip.

At stop 1 we have that

\[
m_1 = m_0 + \frac{w_0}{3} \quad \text{and} \quad w_1 = \frac{2w_0}{3}.
\]

At stop 2 we have that

\[
m_2 = \frac{2m_1}{3} \quad \text{and} \quad w_2 = w_1 + \frac{m_1}{3}.
\]

Given (1), the above expands and simplifies to

\[
m_2 = \frac{2m_0}{3} + \frac{2w_0}{9} \quad \text{and} \quad w_2 = \frac{7w_0}{9} + \frac{m_0}{3}.
\]

Since \( m_2 = w_0 \) it follows from (2) that

\[
w_0 = \frac{2m_0}{3} + \frac{2w_0}{9} \quad \implies \quad m_0 = \frac{7w_0}{6}.
\]

Since \( m_2 + 2 = w_2 \) it follows from (2) that

\[
\frac{2m_0}{3} + \frac{2w_0}{9} + 2 = \frac{7w_0}{9} + \frac{m_0}{3} \quad \implies \quad w_0 = \frac{3m_0 + 18}{5}.
\]
By substituting $m_0 = \frac{7w_0}{6}$ into (3) and solving for $w_0$ we see that

$$w_0 = \frac{3 \left( \frac{7w_0}{6} \right) + 18}{5} \implies w_0 = 12.$$ 

Given $w_0 = 12$ and $m_0 = \frac{7w_0}{6}$ it follows that $m_0 = 14$. Therefore, there are 12 women and 14 men on board the train at the start of the trip.

**CC264.** Kirsty runs three times as fast as she walks. When going to school one day she walks for twice the time she runs and the journey takes 21 minutes. The next day she follows the same route but runs for twice the time she walks. How long does she take to get to school?

*Originally Problem 3 of the 2016–2017 Scottish Mathematical Council Mathematical Challenge, Middle division.*

We received 7 submissions. We present the solution by Titu Zvonaru.

We denote by $d$ the distance to school, $r$ the speed when Kirsty runs, and $w$ the speed when Kirsty walks. We have $r = 3w$. On the first day, Kirsty runs 7 minutes and she walks 14 minutes; this yields $d = 7r + 14w$. Solving with $r = 3w$,

$$d = 35w. \quad (1)$$

Let $x$ be the number of minutes Kirsty walks on the second day. Then she runs $2x$ minutes. It follows that $d = 2xr + xw$, so

$$d = 7xw. \quad (2)$$

By equations 1 and 2 we obtain $x = 5$, hence the second day she takes 15 minutes to get to school.

**CC265.** In the diagram, $ST$ is parallel to $QR$, $UT$ is parallel to $SR$, $PU = 4$ and $US = 6$. Find the length of $SQ$.

*Originally Problem 5 of the 2016–2017 Scottish Mathematical Council Mathematical Challenge, Middle division.*
We received 12 correct submissions. We present the solution by Šefket Arslanagić.

We have $\triangle PUT \sim \triangle PSR$ and from here:
\[
\frac{PU}{PS} = \frac{PT}{PR} \implies \frac{4}{10} = \frac{PT}{PR} \implies \frac{PT}{PR} = \frac{2}{5} \quad (1)
\]

We have too $\triangle PST \sim \triangle PQR$ and from here:
\[
\frac{PS}{PQ} = \frac{PT}{PR}
\]

and from here by equation 1 and $PS = 10$:
\[
\frac{10}{PQ} = \frac{2}{5} \implies PQ = 25.
\]

Now, $SQ = PQ - PS = 25 - 10$, so
\[
SQ = 15.
\]