THE OLYMPIAD CORNER
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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d’une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n’importe quel problème. S’il vous plaît vous référer aux règles de soumission à l’endos de la couverture ou en ligne.

Pour faciliter l’examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 1er avril 2019.

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OC406. Soit $D$ le point à l’intérieur du triangle $ABC$ tel que $BD = CD$ et $\angle BDC = 120^\circ$. Soit $E$ un point à l’extérieur du triangle $ABC$ tel que $AE = CE$, $\angle AEC = 60^\circ$ et les points $B$ et $E$ soient dans les différents demi-plans par rapport à $AC$. Montrer que $\angle AFD = 90^\circ$, où $F$ est le point milieu du segment $BE$.

OC407. Le triangle acutangle isocèle $ABC$ ($AB = AC$) est inscrit dans un cercle de centre $O$. Les rayons $BO$ et $CO$ intersectent les côtés $AC$ et $AB$ aux points $B'$ et $C'$, respectivement. Une droite $l$ est parallèle au segment $AC$ et passe par le point $C'$. Montrer que la droite $l$ est tangente au cercle circonscrit $\omega$ du triangle $B'OC$.

OC408. Est-ce qu’il existe une suite infinie $a_1, a_2, a_3, \ldots$ d’entiers positifs telle que la somme de deux termes distincts de la suite est copremière avec la somme de n’importe quels trois termes distincts de cette suite ?

OC409.

(a) Donner un exemple de fonction continue $f : [0, \infty) \rightarrow \mathbb{R}$ telle que

$$\lim_{x \to \infty} \frac{1}{x^2} \int_0^x f(t) \, dt = 1$$

et $f(x)/x$ n’a pas de limite lorsque $x \to \infty$.

(b) Soit $f : [0, \infty) \rightarrow \mathbb{R}$ une fonction croissante telle que

$$\lim_{x \to \infty} \frac{1}{x^2} \int_0^x f(t) \, dt = 1.$$ 

Montrer que $f(x)/x$ possède une limite lorsque $x \to \infty$ et déterminer cette limite.
OC410. Soit $a_0, a_1, \ldots, a_{10}$ des entiers tels que $a_0 + a_1 + \cdots + a_{10} = 11$. Quel est le nombre maximal de solutions entières distinctes à l’équation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{10} x^{10} = 1.$$ 

OC406. Let $D$ be a point inside the triangle $ABC$ such that $BD = CD$ and $\angle BDC = 120^\circ$. Let $E$ be a point outside the triangle $ABC$ such that $AE = CE$, $\angle AEC = 60^\circ$ and points $B$ and $E$ are in different half-planes with respect to $AC$. Prove that $\angle AFD = 90^\circ$, where $F$ is the midpoint of the segment $BE$.

OC407. The acute isosceles triangle $ABC$ ($AB = AC$) is inscribed in a circle with center $O$. The rays $BO$ and $CO$ intersect the sides $AC$ and $AB$ in the points $B'$ and $C'$, respectively. A line $l$ parallel to the line $AC$ passes through point $C'$. Prove that the line $l$ is tangent to the circumcircle $\omega$ of the triangle $B'OC$.

OC408. Does there exist an infinite increasing sequence $a_1, a_2, a_3, \ldots$ of positive integers such that the sum of any two distinct terms of the sequence is coprime with the sum of any three distinct terms of the sequence?

OC409.

(a) Give an example of a continuous function $f : [0, \infty) \to \mathbb{R}$ such that

$$\lim_{x \to \infty} \frac{1}{x^2} \int_0^x f(t) \, dt = 1$$

and $f(x)/x$ has no limit as $x \to \infty$.

(b) Let $f : [0, \infty) \to \mathbb{R}$ be an increasing function such that

$$\lim_{x \to \infty} \frac{1}{x^2} \int_0^x f(t) \, dt = 1.$$ 

Prove that $f(x)/x$ has a limit as $x \to \infty$ and determine this limit.

OC410. Let $a_0, a_1, \ldots, a_{10}$ be integers such that $a_0 + a_1 + \cdots + a_{10} = 11$. Find the maximum number of distinct integer solutions to the equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{10} x^{10} = 1.$$ 

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OLYMPIAD SOLUTIONS

OC346. Two real number sequences are given, one arithmetic \((a_n)_{n \in \mathbb{N}}\) and another geometric \((g_n)_{n \in \mathbb{N}}\), neither of them constant. These sequences satisfy

\[a_1 = g_1 \neq 0, \quad a_2 = g_2 \quad \text{and} \quad a_{10} = g_3.\]

Prove that, for every positive integer \(p\), there is a positive integer \(m\), such that \(g_p = a_m\).

Originally 2016 Spain Mathematical Olympiad Day 1, Problem 1.

We received 10 solutions. We present the solution by C. R. Pranesachar.

Assume that \(d\) is the common difference of the arithmetic sequence \((a_n)_{n \in \mathbb{N}}\) and \(r\) is the common ratio of the geometric sequence \((g_n)_{n \in \mathbb{N}}\). Then

\[a_2 = a_1 + d = g_2 \quad \text{and} \quad a_{10} = a_1 + 9d = g_3.\]

Since \(g_2^2 = g_1g_3\), we get \((a_1 + d)^2 = a_1(a_1 + 9d)\). This gives \(d = 7a_1\), as \(d \neq 0\) (the sequence \((a_n)_{n \in \mathbb{N}}\) is nonconstant). Furthermore,

\[r = \frac{g_2}{g_1} = \frac{a_1 + d}{a_1} = \frac{8a_1}{a_1} = 8.\]

Hence, if \(p \in \mathbb{N}\), then

\[g_p = a_1r^{p-1} = a_1 \cdot (1 + 7k)\]

for some \(k \in \mathbb{N}\). Taking \(m = k + 1\), we have

\[g_p = a_1 + (m - 1)d = a_m,\]

as desired. This completes the proof.

OC347. Consider the following system of 10 equations in 10 real variables \(v_1, \ldots, v_{10}\):

\[v_i = 1 + \frac{6v_i^2}{v_1^2 + v_2^2 + \cdots + v_{10}^2} \quad (i = 1, \ldots, 10).\]

Find all 10-tuples \((v_1, v_2, \ldots, v_{10})\) that are solutions of this system.

Originally 2016 Canadian Mathematical Olympiad, Problem 2.

We received 4 solutions. We present the solution by Mohammed Aassila.

We prove that there are eleven 10-tuples that are solutions of the system:

\[(4, 4, 4, \ldots, 4) \quad \text{(and all its permutations)} \quad \text{and} \quad (\frac{8}{3}, \frac{8}{3}, \ldots, \frac{8}{3}).\]
Let $C = \sum_{i=1}^{10} v_i^2$. The term $v_i$ is a solution to the equation

$$\frac{6}{C} x^2 - x + 1 = 0.$$  

Therefore, $v_i \in \{a, b\}$. Assume that there are $u$ terms equal to $a$ and $v$ terms equal to $b$. So, $u + v = 10$ and adding all the equations, we get

$$ua + vb = 16. \quad (1)$$

By Vieta’s formulas we have

$$a + b = ab = \frac{C}{6} = \frac{ua^2 + vb^2}{6}.$$  

Plugging $b = \frac{a}{a - 1}$ and $v = 10 - u$ into $(1)$, we get

$$ua^2 - 2a(u + 3) + 16 = 0.$$  

The discriminant of this equation in $a$ is $\Delta_u = (u + 3)^2 - 16u = (u - 1)(u - 9)$. So, $u \leq 1$ or $u \geq 9$. Assume without loss of generality that $u \geq 9$.

(i) If $u = 9$, we have $9a + b = 16$ and $a + b = \frac{9a^2 + b^2}{6}$. The last equation gives $(3a - 2b)^2 = 0$, i.e. $3a = 2b$, which gives the solution $\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \ldots, \frac{4}{3}\right)$ (and all its permutations).

(ii) If $u = 10$, we have $a = 1 + \frac{6a^2}{10a^2} = 1 + \frac{6}{10} = \frac{8}{5}$, so we have the solution $\left(\frac{8}{5}, \frac{8}{5}, \ldots, \frac{8}{5}\right)$.

**OC348.** Triangle $ABC$ is an acute isosceles triangle ($AB = AC$) and $CD$ one altitude. Circle $C_2(C, CD)$ meets $AC$ at $K$, $AC$ produced at $Z$ and circle $C_1(B, BD)$ at $E$. Line $DZ$ meets circle $(C_1)$ at $M$. Show that:

a) $\angle ZDE = 45^\circ$.

b) Points $E, M, K$ lie on a line.

c) $BM \parallel EC$.

*Originally 2016 Greece National Olympiad, Problem 3.*

We received 7 solutions. We present the solution by Oliver Geupel.

Since $CD$ is an altitude in triangle $ABC$, we have $\angle ACD = 90^\circ - \angle A$ and $\angle DCB = 90^\circ - \angle C$. Because $D$ and $E$ are symmetric with respect to the axis $BC$, we have that $\angle BCE = \angle DCB$. Moreover, $\angle B = \angle C$. Hence,

$$\angle ECZ = 180^\circ - (\angle ACD + \angle DCB + \angle BCE) = \angle A + \angle B + \angle C - 90^\circ = 90^\circ.$$  

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Since $D, E, Z$ lie on a circle with centre $C$, we obtain $\angle ZDE = \angle ECZ/2 = 45^\circ$, which is the result a).

Because $D, E, K, Z$ lie on a circle with centre $C$, we have

$$\angle KED = \angle KZD = \frac{1}{2} \angle KCD = \frac{1}{2} \angle ACD = \frac{1}{2} (90^\circ - \angle A) = \angle C - 45^\circ.$$ 

$D, E, M$ lie on a circle with centre $B$. Thus,

$$\angle DME = 180^\circ - \frac{1}{2} \angle EBD = 180^\circ - \angle C.$$ 

Therefore $\angle MED = 180^\circ - \angle EDM - \angle DME = \angle C - 45^\circ = \angle KED$, so that $E, M, K$ lie on a line, which proves result b).

Since $CE$ is tangent to $C_1$, we have $\angle CEB = 90^\circ$. Moreover, because $D, E, M$ lie on a circle with centre $B$, we have $\angle EBM = 2 \angle EDM = 90^\circ$. Consequently, $BM \parallel EC$. This completes the proof of result c).

**OC349.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(yf(x) - x) = f(x)f(y) + 2x$$

for all $x, y \in \mathbb{R}$.

*Originally 2016 Japan Mathematical Olympiad Finals, Problem 4.*

*We received 5 solutions. We present the solution by Oliver Geupel.*

It is straightforward to verify that the two functions $x \mapsto -2x$ and $x \mapsto 1 - x$ are solutions of the problem. We prove that there are no further solutions.

Suppose that $f$ is such a solution. Let $P(x, y)$ denote the assertion that the given functional equation holds for $x$ and $y$. From $P(0, 0)$, we easily obtain $f(0) \in \{0, 1\}$. If $f(0) = 0$, then $P(-x, 0)$ yields $f(x) = -2x$ for all $x \in \mathbb{R}$.

It remains to consider the case $f(0) = 1$. From $P(x, 0)$, we get

$$f(-x) = f(x) + 2x \quad \text{(assertion $Q(x)$)}.$$
Therefore, we see that for all \( x, y \in \mathbb{R} \) such that \( f(x) \neq 0 \),

\[
f(y - x) = f(y + x) + 2x \quad \text{(assertion}\ R(x, y)).
\]

Let \( a \) be a real number with the property that \( f(a) = 0 \). Then, \( a \neq 0 \) and, by \( Q(a) \), \( f(-a) = f(a) + 2a = 2a \neq 0 \), so that \( R(-a, y) \) and thus \( R(a, y) \) does hold. As a consequence, \( R(x, y) \) is satisfied in fact for all \( x, y \in \mathbb{R} \). We deduce

\[
f(y - x) + y - x = f(y + x) + y + x,
\]

that is, \( f(x) + x \) is a constant for all \( x \). Taking account of \( f(0) = 1 \), we conclude \( f(x) = 1 - x \) for all \( x \in \mathbb{R} \). This completes the proof that no further solutions exist.

**OC350.** Two players, \( A \) (first player) and \( B \), take alternate turns in playing a game using 2016 chips as follows: the player whose turn it is, must remove \( s \) chips from the remaining pile of chips, where \( s \in \{2, 4, 5\} \). No one can skip a turn. The player who at some point is unable to make a move (cannot remove chips from the pile) loses the game. Which of the two players has a winning strategy?

*Originally 2016 Philippines Mathematical Olympiad, Problem 4.*

We received 2 solutions and we present both of them.

**Solution 1, by Ivko Dimitrić.**

A player whose turn it is can win if that player is presented at each turn with a pile of chips whose number is between \( 7k + 2 \) and \( 7k + 6 \) for some integer \( k \) and leaves a reduced pile (after removing a suitable number of chips) with number of chips in it equal to \( 7k \) or \( 7k + 1 \) for the next player. Namely, if the number of chips from that range is \( 7k + 2 \) or \( 7k + 3 \), the player removes 2 chips. If the number of chips is \( 7k + 4 \) or \( 7k + 5 \), the player removes 4 chips and if the number is \( 7k + 6 \) the player is to remove 5 chips. That way, the reduced pile now containing \( 7k \) or \( 7k + 1 \) chips is left to the next player at that player’s turn. However many chips \( s \in \{2, 4, 5\} \) the next player removes, the new pile that remains have at least \( 7k - 5 = 7(k - 1) + 2 \) and at most \( (7k + 1) - 2 = 7(k - 1) + 6 \) chips, leaving it in the desirable range for the player who plays subsequently.

Since 2016 = 288 \cdot 7 \) is a multiple of 7 at the beginning of the game, however many chips \( s \in \{2, 4, 5\} \) player \( A \) removes, it would leave a new pile whose number is in the range \( 287 \cdot 7 + 2 = 2011 \) to \( 287 \cdot 7 + 6 = 2015 \) for player \( B \). Then player \( B \) removes a certain number of chips following the strategy as explained above so that the pile left after \( B \) makes the move has between \( 287 \cdot 7 \) and \( 287 \cdot 7 + 1 \) chips.
at player $A$’s turn. Whatever move the player $A$ next does, the new reduced pile left will have a number of chips between $286 \cdot 7 + 2$ and $286 \cdot 7 + 6$ at $B$’s turn. Consequently, in order to win, following the strategy, the player $B$ ensures that the number of chips left after $B$’s turn is between $285 \cdot 7$ and $285 \cdot 7 + 1$, and so on, resulting in $B$ leaving always a number of chips between $7k$ and $7k + 1$, so that after $A$’s subsequent move $B$ has a pile numbering between $7(k-1) + 2$ and $7(k-1) + 6$ chips at $B$’s disposal. Thus, eventually, as $k$ decreases to 1, player $B$, who follows the strategy, will leave the pile reduced to only 7 or 8 chips. If that number is 7 and player $A$ next takes 2, 4 or 5 chips (leaving a new pile of 5, 3 or 2 chips, respectively), player $B$ will remove 4, 2 or 2 chips, respectively, so that $A$ cannot make the next move. Likewise, if the number of chips remaining is 8 and player $A$ removes 2, 4 or 5 chips (leaving the reduced pile of 6, 4 or 3 chips) then player $B$ removes 5, 4 or 2 chips, respectively, so that $A$ cannot continue the play according to the rules and player $B$ wins.

Solution 2, by Missouri State University Problem Solving Group.

More generally, if the game starts with $n$ chips, we claim that $B$ has a winning strategy if $n \equiv 0$ or $1 \mod 7$ and $A$ has a winning strategy otherwise. We proceed by induction on $n$. If $n = 0$ or 1, $A$ clearly loses. If $n = 2, 4$ or 5 $A$ picks up all the chips and wins. If $n = 3$, $A$ picks up 2 chips and $B$ loses. If $n = 6$, $A$ picks up 5 chips and $B$ loses. Now suppose $n > 6$ and the result holds for all $k < n$. Note that if at any point in the game, removing 2, 4, or 5 chips always results in a number of chips where $A$ has a winning strategy at the beginning of the game, then the player whose turn is next has a winning strategy. If at any point in the game, some choice of removing 2, 4, or 5 chips results in a number of chips where $B$ has a winning strategy at the beginning of the game, then the player whose turn it is has a winning strategy.

Suppose $n \equiv 0 \mod 7$. Removing 2, 4 or 5 chips leaves $k$ chips where $k \equiv 2, 3$ or $5 \mod 7$. By induction, these all yield a winning strategy for $A$, so by discussion above, since $A$ is starting, this gives a winning strategy for player $B$.

Suppose $n \equiv 1 \mod 7$. Removing 2, 4, or 5 chips leaves $k$ chips where $k \equiv 3, 4$, or $6 \mod 7$. By induction, these all yield a winning strategy for $A$, so again $B$ has a winning strategy.

Suppose $n \equiv 2 \mod 7$. If $A$ removes 2 chips we have $k \equiv 0 \mod 7$ chips remaining. By induction, this number gives a winning strategy for $B$, so the original number of chips gives a winning strategy for $A$.

Proceeding in a similar manner if $n \equiv 3 \mod 7$, $A$ removes 2 chips, if $n \equiv 4 \mod 7$, $A$ removes 4 chips, if $n \equiv 5 \mod 7$, $A$ removes 5 chips, and if $n \equiv 6 \mod 7$, $A$ removes 5 chips. In each case the number of remaining chips is $k \equiv 0$ or $1 \mod 7$. Consequently $A$ has a winning strategy in each of these cases.

If the initial number of chips is $2016 \equiv 0 \mod 7$, then $B$ has a winning strategy.

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