CONTEST CORNER
SOLUTIONS


CC296. Find the number of positive integers $k$ with $10000 \leq k \leq 99999$ such that the middle digit is the average of the first and fifth digits.

Originally problem 22 from the University of Vermont, 56th Annual High School Prize Examination, 2013.

We received 8 submissions, of which 4 were correct and complete and others contain a variety of counting mistakes. We present two of the solutions.

Solution 1, by Henry Ricardo.

Consider the number $k = \overline{abcde}$, where $10000 \leq k \leq 99999$. If digit $c$ is to be the average of digits $a$ and $e$, then $a + e$ must be even, so that both $a$ and $e$ are even or both $a$ and $e$ are odd.

Case 1: Both $a$ and $e$ are even. Since $a$ can’t equal zero, we have 4 choices for $a$ and 5 choices for $e$. Any values for $a$ and $e$ determine $c$ uniquely. After selecting $a$, $e$, and $c$, we have 10 choices for $b$ and 10 choices for $d$, giving us $4 \times 10 \times 1 \times 10 \times 5 = 2000$ numbers satisfying our condition.

Case 2: Both $a$ and $e$ are odd. Now we have 5 choices for $a$, after which there are 5 ways to choose $e$. With $a$, $e$, and therefore $c$ determined, we have 10 choices for $b$ and 10 choices for $d$. Thus we have $5 \times 10 \times 1 \times 10 \times 5 = 2500$ numbers satisfying our condition.

This analysis yields a total of 4500 positive integers $k$ with $10000 \leq k \leq 99999$ such that the middle digit is the average of the first and fifth digits.

Solution 2, by Doddy Kastanya.

Each integer of interest will have 5 digits and can be represented by $AxByC$, where $B = \frac{1}{2}(A + C)$, where $A$ could be any integer from 1 through 9 and $C$ could be any integer between 0 and 9. There are 45 feasible triples $(A,B,C)$ as indicated by the shaded cells in the table on the next page.

The other two digits of the integer (i.e., $x$ and $y$) are independent of the values of $A$, $B$, or $C$. Since there are 10 possible values of $x$ and 10 possible values of $y$, there are 100 different permutations of $x$ and $y$. Therefore, there are $45 \times 100 = 4500$ positive integers $k$ with $10000 \leq k \leq 99999$ such that the middle digit is the average of the first and fifth digits.

Cruix Mathematicorum, Vol. 44(10), December 2018
Six black checkers are placed on squares of a 6 by 6 checkerboard in the positions shown in Figure 1 and are left in place. A white checker begins on the square at the lower left corner of the board (marked A in Figure 1) and follows a path from square to square across the board, ending in the upper right corner of the board (marked B). How many different paths are there from A to B if at each step the white checker can move one square to the right, one square up or one square diagonally upward to the right and may not pass though any square occupied by a black checker? One such path is shown in Figure 2.

\[
\begin{array}{cccccccccc}
\text{A} & & & & & & & & & \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 1 & 2 & 3 & 4 & 5 & & & & \\
2 & 1 & 2 & 3 & 4 & 5 & & & & \\
3 & & 2 & 3 & 4 & 5 & 6 & & & \\
4 & 2 & & 3 & 4 & 5 & 6 & & & \\
5 & & 3 & 4 & 5 & 6 & 7 & & & \\
6 & & & 4 & 5 & 6 & 7 & & & \\
7 & & & & 5 & 6 & 7 & & & \\
8 & 4 & & & 5 & 6 & 7 & & & \\
9 & 5 & & & & 6 & 7 & 8 & 9 & \\
\end{array}
\]

**CC297.** Six black checkers are placed on squares of a 6 by 6 checkerboard in the positions shown in Figure 1 and are left in place. A white checker begins on the square at the lower left corner of the board (marked A in Figure 1) and follows a path from square to square across the board, ending in the upper right corner of the board (marked B). How many different paths are there from A to B if at each step the white checker can move one square to the right, one square up or one square diagonally upward to the right and may not pass though any square occupied by a black checker? One such path is shown in Figure 2.

Originally problem 27 from the University of Vermont, 56th Annual High School Prize Examination, 2013.

We received 4 submissions, of which 2 were correct and complete. We present the solution by Carlos Moreno and Ángel Plaza.

Let us assign integer coordinates to the cells in our checkerboard, beginning with \(A = (0, 0)\), so \(B = (5, 5)\). Let \(D(n, k)\) be the number of different paths from \(A\) to the cell \((n, k)\). The following recurrence relation holds:

\[
D(n, k) = \begin{cases} 
0, & \text{if there is a black checker in it. Otherwise,} \\
1, & \text{if } n = 0 \text{ or } k = 0 \\
D(n - 1, k) + D(n - 1, k - 1) + D(n, k - 1), & \text{otherwise.}
\end{cases}
\]
Using the recurrence relation in our problem gives the numbers in each cell as in the following figure:

\[
\begin{array}{cccccc}
1 & 2 & 9 & 16 & 42 & 122 \\
1 & 7 & 13 & 0 & 2 & \ \\
1 & 5 & 13 & 0 & 2 & \ \\
1 & 3 & 5 & 0 & 2 & \ \\
1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]

Therefore, there are 122 different paths from A to B.

Note: this problem may be solved using the Delannoy numbers.

**CC298.** Let \( ABC \) be the triangle with vertices \((0, 0), (4, 0)\) and \((2, 3)\). Find the coordinates of the point \( P \) that is equidistant from \( A, B \) and \( C \).

*Originally problem 36 from the University of Vermont, 56th Annual High School Prize Examination, 2013.*

We received 9 solutions. We present 2 solutions, slightly modified by the editor.

**Solution 1, by Catherine Doan.**

Let \((x, y)\) be the coordinates of \( P \). Let \( P_A, P_B, P_C \) denote the distances from point \( P \) to the points \( A, B, C \), respectively. As \( P_A = P_B \), it follows that \( x = 2 \).

As \( P_A = P_C \) and \( x = 2 \), it follows that \( 4 + y^2 = (y - 3)^2 \), so \( y = \frac{5}{6} \). Therefore the coordinates of \( P \) are \((2, \frac{5}{6})\).

**Solution 2, by Ivko Dimitrić.**

Since the point \( P \) is the circumcenter of an isosceles \( \triangle ABC \), we use the well known formula for the circumradius \( R \) of a triangle in terms of its area \([ABC]\) and side lengths \(a, b, c\) to get

\[
R = \frac{abc}{4[ABC]} = \frac{4 \times \sqrt{2^2 + 3^2}^2}{4 \times \frac{1}{2} \times 4 \times 3} = \frac{13}{6}.
\]

The \( y \)-coordinate of \( P \) is \( 3 - \frac{13}{6} = \frac{5}{6} \). Thus the coordinates of \( P \) are \((2, \frac{5}{6})\).

**CC299.** Find the area of the region bounded by the graphs of

\[
\begin{cases}
x + y + |x| = 10, \\
x + y - |x| = -8.
\end{cases}
\]

*Originally problem 24 from the University of Vermont, 56th Annual High School Prize Examination, 2013.*

*Cruix Mathematicorum, Vol. 44(10), December 2018*
We received 8 correct solutions and 1 incorrect submission. We present the solution by Kathleen Lewis.

The function \(x + y + |x| = 10\) is equal to \(y = 10\) if \(x < 0\) and \(y = 10 - 2x\) when \(x \geq 0\). The function \(x + y - |x| = -8\) is equal to \(y = -8\) if \(x \geq 0\) and \(y = -2x - 8\) if \(x < 0\). The functions are equal when \(|x| = 9\), so \(x = \pm 9\). The enclosed area is a parallelogram with base length 9 and height 18. Therefore the area is 162.

**CC300.** Three poles with circular cross sections are to be bound together with a wire. The radii of the circular cross sections are 1, 3 and 1 inches. The centers of the circles are on the same straight line as indicated in the sketch. If the length of the wire is written in the form \(a\sqrt{3} + b\pi\), where \(a\) and \(b\) are rational numbers, find \(a + b\). Assume that the wire has negligible thickness.

*Originally problem 41 from the University of Vermont, 56th Annual High School Prize Examination, 2013.*

We received 5 solutions. We present the solution by Catherine Doan.

Let \(A, B, C\) be the centers of the three circles, i.e. the three circular cross sections. Let \(E, F\) be the tangent points of the wires to the circles with \(A\) and \(B\) centers as shown in the above diagram. We have \(EF \perp AE, EF \perp BF\). Let \(H\) be on \(EA\) such that \(BH \parallel EH\).

Due to ratio of side to hypotenuse being \(1 : 2\), triangle \(ABH\) is a 30-60-90 right triangle, hence \(EF = BH = 2\sqrt{3}\). Since \(\angle FBF' = 120^\circ\), the length of \(FF'\) arc (shown in purple) is \((2\pi \times 1)/3 = 2\pi/3\). Similarly because \(\angle EAG = 60^\circ\), the length of \(EG\) arc (shown in green) is \((2\pi \times 3)/6 = \pi\). Due to symmetry, the total length of the wire is:

\[
2 \times \frac{2\pi}{3} + 2\pi + 4 \times 2\sqrt{3} = 8\sqrt{3} + \frac{10}{3}\pi.
\]

The answer is: \(a = 8\), \(b = 10/3\), and \(a + b = 8 + 10/3 = 34/3\).