INTRODUCTION

In this number, we continue our selection of relations in the triangle, focusing on formulas involving lengths related to the classical cevians. As in part I, the notations are standard and borrowed from [2].

ABOUT THE ALTITUDES $h_a, h_b, h_c$

Besides the easy

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

(which readily follows from $\frac{1}{h_a} = \frac{a}{2F} = \frac{a}{2R}$ and similar relations), we consider a less known, easy-to-remember formula:

$$\left(\frac{h_a + h_b + h_c}{h_a} \cdot \frac{1}{h_b} + \frac{1}{h_c}\right) = (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$  \hspace{1cm} (1)

(again an obvious consequence of $2F = ah_a = bh_b = ch_c$).

With the help of the identities

$$(x + y + z)(xy + yz + zx) - 3xyz = \sum_{\text{cyclic}} x^2(y + z) = (x + y)(y + z)(z + x) - 2xyz, \hspace{1cm} (2)$$

we may equivalently write (1) as

$$\frac{(h_a + h_b)(h_b + h_c)(h_c + h_a)}{h_ah_bh_c} = \frac{(a + b)(b + c)(c + a)}{abc}$$  \hspace{1cm} (3)

and give a solution to problem 3453 [2009 : 325,328 ; 2010 : 342] that asked for the inequality

$$8 \left(\sum_{\text{cyclic}} h_a^2(h_b + h_c)\right) + 16h_ah_bh_c \leq 3\sqrt{3} \left(\sum_{\text{cyclic}} a^2(b + c)\right) + 6\sqrt{3}abc.$$

Indeed, a consequence of (2) is that this inequality is equivalent to $Q \leq \frac{3\sqrt{3}}{8}$ where

$$Q = \frac{(h_a + h_b)(h_b + h_c)(h_c + h_a)}{(a + b)(b + c)(c + a)}.$$

But from (3) we have

$$Q = \frac{h_ah_bh_c}{abc} = \frac{8F^3}{(abc)^2} = \frac{abc}{8R^3} = \sin A \sin B \sin C,$$
hence, using AM-GM and the concavity of the Sine function on $(0, \pi)$,
\[
Q \leq \left( \frac{\sin A + \sin B + \sin C}{3} \right)^3 \leq \left( \sin \left( \frac{A + B + C}{3} \right) \right)^3 = \frac{3\sqrt{3}}{8}.
\]

**About the distances $IA, IB, IC$**

Prompted by the intervention of $IA$ in part I, we now present a couple of interesting relations connecting the distances $IA, IB, IC$ to other elements of the triangle.

First we consider the product $IA \cdot IB \cdot IC$ and show that
\[
sIA \cdot IB \cdot IC = r \cdot abc.
\]  
(4)

The proof is easy:
\[
IA \cdot IB \cdot IC = \frac{r}{\sin \frac{A}{2}} \cdot \frac{r}{\sin \frac{B}{2}} \cdot \frac{r}{\sin \frac{C}{2}} = \frac{r^3}{2R} = \frac{r}{s} \cdot abc.
\]

At this point, it is worth mentioning a beautiful formula that also involves the excenters $I_a, I_b, I_c$:
\[
IA \cdot IB \cdot IC \cdot I_aA \cdot I_bB \cdot I_cC = (abc)^2.
\]  
(5)

To see this, we first remark that
\[
bc \cos^2 \frac{A}{2} = \frac{bc(1 + \cos A)}{2} = \frac{bc}{2} \left( 1 + \frac{b^2 + c^2 - a^2}{2bc} \right) = s(s - a),
\]  
(6)

from which we deduce that
\[
IA \cdot I_aA = \frac{s - a}{\cos \frac{A}{2}} \cdot \frac{s}{\cos \frac{A}{2}} = bc.
\]

Similarly, $IB \cdot I_bB = ca, IC \cdot I_cC = ab$ and (5) follows.

Relation (5) reminds us of the known relation $w_aw_bw_hw_aw_whc = (abc)^2$ where $W_a, W_b, W_c$ denote the lengths of the angle bisectors extended until they are chords of the circumcircle (see problem 168 [1976 : 136 ; 1977 : 233]). It is interesting to notice that we even have $w_bw_a = bc = IA \cdot I_aA$ and similar relations.

We conclude this paragraph with the formula
\[
aIA^2 + bIB^2 + cIC^2 = abc,
\]  
(7)

from which we will derive a general inequality.

Using (6), we obtain
\[
aIA^2 + bIB^2 + cIC^2 &= a \left( \frac{s - a}{\cos \frac{A}{2}} \right)^2 + b \left( \frac{s - b}{\cos \frac{B}{2}} \right)^2 + c \left( \frac{s - c}{\cos \frac{C}{2}} \right)^2 \\
&= a(s - a)^2 \frac{bc}{s(s - a)} + b(s - b)^2 \frac{ca}{s(s - b)} + c(s - c)^2 \frac{ab}{s(s - c)} \\
&= \frac{abc}{s} (s - a + s - b + s - c)
\]
and (7) follows.

A nice application is the inequality

\[ aIA \cdot PA + bIB \cdot PB + cIC \cdot PC \geq abc \]

that holds for any point \( P \) in the plane of the triangle \( ABC \). To prove it, we use the dot product and the Cauchy-Schwarz inequality as follows:

\[
aPA \cdot IA + bPB \cdot IB + cPC \cdot IC \\
= \|aIA\| \|IA - IP\| + \|bIB\| \|IB - IP\| + \|cIC\| \|IC - IP\| \\
\geq aIA \cdot (IA - IP) + bIB \cdot (IB - IP) + cIC \cdot (IC - IP) \\
= aIA^2 + bIB^2 + cIC^2 - IP \cdot (aIA + bIB + cIC) \\
= abc
\]

(since \( aIA + bIB + cIC = \vec{0} \)).

**About the exradii** \( r_a, r_b, r_c \)

Faced with the proof of a relation between the exradii \( r_a, r_b, r_c \), the first move is often to use the equalities

\[ F = r_a(s - a) = r_b(s - b) = r_c(s - c). \] (8)

For examples, the striking formulas

\[
\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}, \quad r_ar_b + r_br_c + r_cr_a = s^2 = \frac{r_ar_br_c}{r}
\]

and

\[
\sqrt{\frac{r_ar_br_c}{r_a}} + \sqrt{\frac{r_br_cr_a}{r_b}} + \sqrt{\frac{r_ar_br_c}{r_c}} = s
\]

are straightforwardly deduced from (8) and \( F = rs = \sqrt{s(s-a)(s-b)(s-c)} \).

With the additional known formulas

\[ ab + bc + ca = s^2 + r^2 + 4rR \quad \text{and} \quad a^2 + b^2 + c^2 = 2s^2 - 2r^2 - 8rR, \]

we easily obtain

\[ r_a + r_b + r_c = r + 4R \quad \text{and} \quad r_a^2 + r_b^2 + r_c^2 + a^2 + b^2 + c^2 = 16R^2 \]

that were at work in problem 3570 [2010 : 397,399 ; 2011 : 402].

Since \( 2F = ah_a = bh_b = ch_c \), one can expect some connections with \( h_a, h_b, h_c \). A good example is

\[
\frac{h_b + h_c}{r_a} + \frac{h_b + h_c}{r_a} + \frac{h_b + h_c}{r_a} = 6 \] (9)
which is mentioned but not proved in [1]. Here is a quick proof. Since
\[ h_b + h_c = 2F \left( \frac{1}{b} + \frac{1}{c} \right) = \frac{2F(ab + ac)}{abc}, \]
the left-hand side of (9) rewrites as
\[ \frac{2}{abc} ((ab + ac)(s - a) + (bc + ba)(s - b) + (ca + cb)(s - c)) = \frac{2}{abc} - (ab(c) + bc(a) + ca(b)) \]
and (9) follows.

The reader will find other formulas of the same kind in exercise 1.

**A mixed formula**

A long time ago, I came across the following impressive formula in an old copy of the 1886 *Journal de mathématiques élémentaires* Vuibert,

\[ \frac{w_a^2}{h_a} \sqrt{\frac{m_a^2 - h_a^2}{w_a^2 - h_a^2}} = 2R. \]  
(10)

(Of course, a similar result holds if the subscript \( a \) is replaced by \( b \) or \( c \).) This formula was given with a typo (\( r \) instead of \( R \)) and without proof!

A possible proof is as follows. With the help of the known formulas
\[ w_a^2 = bc(a + b + c)(b + c - a) \]
\[ h_a = \frac{2F}{a} = \frac{bc}{2R} \]
\[ 4m_a^2 = 2b^2 + 2c^2 - a^2 \]
and
\[ 16F^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4) = (a + b + c)(b + c - a)(c + a - b)(a + b - c), \]
we first obtain
\[ w_a^2(4m_a^2 - 4h_a^2) = \frac{b^2c^2}{(b + c)^2} (a + b + c)^2(b + c - a)^2 \left( 2b^2 + 2c^2 - a^2 - \frac{16F^2}{a^2} \right) \]
\[ = \frac{b^2c^2(a + b + c)^2(b + c - a)^2}{a^2(b + c)^2} \]
and, second,
\[ 16R^2h_a^2(w_a^2 - h_a^2) = 4b^2c^2 \left( \frac{bc(a + b + c)(b + c - a)}{(b + c)^2} - \frac{16F^2}{4a^2} \right) \]
\[ = \frac{b^2c^2(a + b + c)(b + c - a)}{a^2(b + c)^2} \left( 4a^2bc - (b + c)^2(c + a - b)(a + b - c) \right). \]

Then (10) follows from \( 4a^2bc - (b + c)^2(c + a - b)(a + b - c) = (b - c)^2(a + b + c)(b + c - a) \) (as it is readily checked). (Variants of proofs can be found in [3].)

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A new youth was recently granted to this relation through several problems composed by Panagiote Ligouras. A typical example is problem 1847 posed in *Mathematics Magazine* in June 2010. Here is the slightly arranged statement:

Prove that in a scalene triangle the following inequality holds

\[
\frac{w_a^4(m_a^2 - h_a^2)}{h_a^2r_a(u_a^2 - h_a^2)} + \frac{w_b^4(m_b^2 - h_b^2)}{h_b^2r_b(u_b^2 - h_b^2)} + \frac{w_c^4(m_c^2 - h_c^2)}{h_c^2r_c(u_c^2 - h_c^2)} > \frac{16}{3}.
\]

Formula (10) allows a quick proof by immediately transforming the required inequality into

\[
\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} > \frac{4r(a + b + c)}{3R^2}.
\]

(11)

From exercise 1 below, we deduce

\[
\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} = \frac{4(4R + r)}{a + b + c}
\]

so that (11) is equivalent to \(3R^2(4R + r) > r(a + b + c)^2\). The proof is easily completed by recalling that \(R > 2r\) and \(a + b + c < 3\sqrt{3}R\).

**Exercises**

1. Prove the formulas

\[
\left(\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c}\right)\left(\frac{a + b + c}{r_a + r_b + r_c}\right) = 4
\]

and

\[
\frac{1}{rr_br_c} + \frac{1}{rr_cr_a} + \frac{1}{rr_ar_b} = \frac{8}{r_ar_br_c}.
\]

2. (from *College Math. Journal* Problem 937) Prove that in a scalene triangle, the following inequality holds

\[
\frac{w_a^4(m_a^2 - h_a^2)}{h_a^2(u_a - h_a)\sqrt{u_a \cdot h_a}} + \frac{w_b^4(m_b^2 - h_b^2)}{h_b^2(u_b - h_b)\sqrt{u_b \cdot h_b}} + \frac{w_c^4(m_c^2 - h_c^2)}{h_c^2(u_c - h_c)\sqrt{u_c \cdot h_c}} \geq 24R^2.
\]

**References**

