OC346. On considère une suite arithmétique non constante \((a_n)_{n \in \mathbb{N}}\) de nombres réels et une suite géométrique non constante \((g_n)_{n \in \mathbb{N}}\) de nombres réels, de manière que \(a_1 = g_1 \neq 0\), \(a_2 = g_2\) et \(a_{10} = g_3\). Démontrer que pour tout entier strictement positif \(p\), il existe un entier strictement positif \(m\) tel que \(g_p = a_m\).

OC347. On considère le système suivant de 10 équations en 10 variables réelles \(v_1, \ldots, v_{10}\):

\[
v_i = 1 + \frac{6v_i^2}{v_1 + v_2 + \cdots + v_{10}} \quad (i = 1, \ldots, 10).
\]

Déterminer tous les 10-uplets \((v_1, v_2, \ldots, v_{10})\) qui sont les solutions du système.

OC348. On considère un triangle acutangle isocèle \(ABC\) \((AB = AC)\) et une hauteur \(CD\) du triangle. Soit \(C_1\) le cercle de centre \(B\) et de rayon \(BD\) et \(C_2\) le cercle de centre \(C\) et de rayon \(CD\). Sachant que \(C_2\) coupe \(AC\) en \(K\), le prolongement de \(AC\) en \(Z\) et \(C_1\) en \(E\) et que la droite \(DZ\) coupe \(C_1\) en \(M\), démontrer que:

a) \(\overline{ZDE} = 45^\circ\),

b) les points \(E, M\) et \(K\) sont alignés,

c) \(BM \parallel EC\).

OC349. Déterminer toutes les fonctions \(f \left( f : \mathbb{R} \to \mathbb{R} \right)\) telles que

\[
f(yf(x) - x) = f(x)f(y) + 2x
\]

pour tous réels \(x\) et \(y\).

OC350. Deux joueurs, \(A\) et \(B\), jouent à un jeu à tour de rôle en commençant par \(A\). Il y a 2016 jetons au départ et le joueur dont c’est le tour doit retirer \(s\)
jetons de la pile, \( s \in \{2, 4, 5\} \). Le joueur qui ne peut plus jouer (c.-à-d. qui ne peut pas retirer un nombre réglementaire de jetons) perd la partie. Lequel des deux joueurs peut avoir une stratégie gagnante?

**OC346.** Two real number sequences are given, one arithmetic \((a_n)_{n \in \mathbb{N}}\) and another geometric \((g_n)_{n \in \mathbb{N}}\), neither of them constant. These sequences satisfy \(a_1 = g_1 \neq 0\), \(a_2 = g_2\) and \(a_{10} = g_3\). Prove that, for every positive integer \(p\), there is a positive integer \(m\), such that \(g_p = a_m\).

**OC347.** Consider the following system of 10 equations in 10 real variables \(v_1,\ldots,v_{10}\):

\[
v_i = 1 + \frac{6v_i^2}{v_1^2 + v_2^2 + \cdots + v_{10}^2} \quad (i = 1,\ldots,10).
\]

Find all 10-tuples \((v_1,v_2,\ldots,v_{10})\) that are solutions of this system.

**OC348.** Triangle \(ABC\) is an acute isosceles triangle \((AB = AC)\) and \(CD\) one altitude. Circle \(C_2(C,CD)\) meets \(AC\) at \(K\), \(AC\) produced at \(Z\) and circle \(C_1(B,BD)\) at \(E\). Line \(DZ\) meets circle \((C_1)\) at \(M\). Show that:

a) \(\overline{ZDE} = 45^\circ\).

b) Points \(E, M, K\) lie on a line.

c) \(BM \parallel EC\).

**OC349.** Find all functions \(f : \mathbb{R} \to \mathbb{R}\) such that

\[
f(yf(x) - x) = f(x)f(y) + 2x
\]

for all \(x, y \in \mathbb{R}\).

**OC350.** Two players, \(A\) (first player) and \(B\), take alternate turns in playing a game using 2016 chips as follows: the player whose turn it is, must remove \(s\) chips from the remaining pile of chips, where \(s \in \{2, 4, 5\}\). No one can skip a turn. The player who at some point is unable to make a move (cannot remove chips from the pile) loses the game. Which of the two players has a winning strategy?
OC286. There are four basketball players $A, B, C, D$. Initially the ball is with $A$. The ball is always passed from one person to a different person. In how many ways can the ball come back to $A$ after seven moves? (For example, $A$ passes to $C$ who passes to $B$ who passes to $D$ who passes to $A$ who passes to $B$ who passes to $C$ who passes to $A$.)

Originally problem 4 of the 2015 India National Olympiad.

We received 4 correct solutions and 1 incorrect submission. We present the solution by Gabriel Wallace.

We claim there are 546 ways. First, we examine the range of $A$. It is trivial to show that the minimum number of times person $A$ has the ball is 2. Since no one can pass the ball to themselves, there has to be a space between each person, so the max number of times for $A$ is 4. Then we have three cases.

Case 1: $A = 4$. Here we have three possibilities, as follows:

\[
\begin{align*}
A & \quad A \quad A \quad A \\
A & \quad - \quad A \quad - \quad A \\
A & \quad - \quad A \quad - \quad A
\end{align*}
\]

A blank space directly following an $A$ can take 3 values: $B$, $C$, or $D$. A space after that would have two possibilities, whatever two letters that were not selected prior. So a 1-blank will have 3 possibilities and a 2-blank will have 6. Notice in all 3 permutations, there are two 1-blanks and one 2-blank. Summing everything, we have $3(3 \cdot 3 \cdot 3 \cdot 2) = 162$ ways for 4 $A$’s.

Case 2: $A = 3$. Here we have four possibilities, as follows:

\[
\begin{align*}
A & \quad A \quad - \quad - \quad - \quad A \\
A & \quad - \quad A \quad - \quad - \quad A \\
A & \quad - \quad - \quad A \quad - \quad - \quad A \\
A & \quad - \quad - \quad - \quad A \quad - \quad A
\end{align*}
\]

Directly following an $A$ we still have 3 possible values. Since we have already fixed the $A$’s, then any other space will have two possible values, whatever was not selected directly before. So here we have two lines with one 1-blank and one 4-blank, and two lines with one 2-blank and one 3 blank. Thus we have $2(3 \cdot 3 \cdot 2^4) + 2(3 \cdot 2 \cdot 3 \cdot 2^2) = 288$ ways for 3 $A$’s.

Case 3: $A = 2$. Here we have one possibility, as follows:

\[
A \quad - \quad - \quad - \quad - \quad - \quad A
\]
Following the same logic as above, for this one 6-blank we have $3 \cdot 2^5 = 96$ ways for 2 $A$’s.

Summing all the cases we have a total of $162 + 288 + 96 = 546$ ways.

**OC287.** Let

$$P(x) = ax^3 + (b - a)x^2 - (c + b)x + c$$

and

$$Q(x) = x^4 + (b - 1)x^3 + (a - b)x^2 - (c + a)x + c$$

be polynomials of $x$ with $a,b,c$ non-zero real numbers and $b > 0$. If $P(x)$ has three distinct real roots $x_0, x_1, x_2$ which are also roots of $Q(x)$, then:

1. Prove that $abc > 28$.
2. If $a, b, c$ are non-zero integers with $b > 0$, find all their possible values.

*Originally problem 2 of the 2015 Greece National Olympiad.*

We received 4 correct submissions. We present the solution by Ali Adnan.

Clearly the fourth root of $Q(x)$ must also be real. Let that root be $x_3$. Then from Viete’s relations,

$$x_0x_1x_2 = -\frac{c}{a}, \quad x_0x_1x_2x_3 = c,$$

$$x_0 + x_1 + x_2 = 1 - \frac{b}{a}, \quad x_0 + x_1 + x_2 + x_3 = 1 - b.$$

From the first set of equations, $x_3 = -a$ and using this in the second set, we get $\frac{b}{a} = b - a$. Again from Viete’s relation

$$x_0x_1 + x_1x_2 + x_2x_0 = -\frac{b + c}{a},$$

$$x_3(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_2x_0 = a - b.$$

The above two relations imply that

$$(-a) \cdot \left(1 - \frac{b}{a}\right) + \frac{- (b + c)}{a} = a - b \Rightarrow \frac{b}{a} + \frac{c}{a} = 2(b - a) \Rightarrow c = b.$$

So $abc = ab^2$, while

$$\frac{b}{a} = b - a \Rightarrow b = \frac{a^2}{a - 1}.$$

(Note that the above relation implies that $a > 1$).

Thus $ab^2 = a^3(a - 1)^{-2} (= f(a)$, say). It is routine calculus to obtain $f'(a) = 0 \Rightarrow a = 5/3$ and check that $a = 5/3$ gives a minimum indeed for $f(a), a > 1$. After that it is easily obtained that $f(5/3) > 28$, which proves part 1.
For part 2, it is seen that \( b = \frac{a^2}{a - 1} \) implies that
\[
(a - 1)|a^2 \Rightarrow (a - 1)|(a^2 - (a^2 - 1)) \Rightarrow a - 1 = \pm 1.
\]
But we also see that \( a > 1 \) and so \( a - 1 = 1 \Rightarrow a = 2 \). Thus \( b = c = 4 \). So in summary the only possible triplet of non-zero integral values of \((a, b, c)\) with \( b > 0 \) is \((2, 4, 4)\).

\textbf{OC288}. Find all positive integers \( n \) such that for any positive integer \( a \) relatively prime to \( n \), \( 2n^2 \mid a^n - 1 \).

Originally problem 6 from day 2 of the 2015 Turkey National Olympiad.

We present the solution by Steven Chow. There were no other submissions.

If \( n \equiv 1 \pmod{2} \), then since there are infinitely many primes, there exists integer \( a \) relatively prime to \( n \) such that \( a \equiv 0 \pmod{2} \), so \( 2n^2 \nmid a^n - 1 \), which is a contradiction.

Let \( n_1 \) be the integer such that \( 2n_1 = n \). Therefore
\[
2n^2 \mid a^n - 1 \implies 2(2n_1)^2 \mid a^{2n_1} - 1 \implies 2^3 n_1^2 \mid (a^{n_1} + 1)(a^{n_1} - 1).
\]

Let \( 2^k \mid n_1 \). From Dirichlet’s Theorem, there exists positive integer \( a \) relatively prime to \( n = 2n_1 \) such that \( a \equiv 5 \pmod{8} \), so
\[
2^3 n_1^2 \mid (a^{n_1} + 1)(a^{n_1} - 1) = \left( \prod_{j=0}^{k} \left( a^{\frac{n_1}{2^j}} + 1 \right) \right) \left( a^{\frac{n_1}{2^k}} - 1 \right)
\]
\[
\implies 3 + 2k \leq (k + 1) + 2 \implies k \leq 0.
\]

Therefore \( k = 0 \).

If \( n_1 \) is not divisible by a prime, then \( n_1 = 1 \) and \( n = 2 \). For all positive integers \( a \) relatively prime to \( 2 = n \), \( 2n^2 = 2(2)^2 \mid a^2 - 1 = a^n - 1 \), so this works.

Otherwise, \( n_1 \) is divisible by a prime.

If there exists a prime \( p \geq 3 \) and an integer \( m \) such that \( p^m \mid n_1 \), then
\[
p^{2m} \mid n_1^2 \mid a^{2n_1} - 1 \implies a^{2n_1} \equiv 1 \pmod{p^{2m}},
\]
so since \( p^{2m} \) has primitive roots (well known), we have
\[
p^{2m-1} \mid p^{2m-1}(p - 1) = \phi(p^{2m}) \mid 2n_1 \implies 2m - 1 \leq m \implies m \leq 1.
\]

Therefore \( n \) is square free.

Let \( p_1 \) be the least prime such that \( p_1 \mid n_1 \). Since \( k = 0 \), \( p_1 \geq 3 \). By definition, \( p_1 - 1 \nmid n_1 \), and from above, \( p_1(p_1 - 1) \mid 2n_1 \), so \( p_1 = 3 \).

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If \( n = (2)(3) \), then from Euler’s Theorem, for any positive integer \( a \) relatively prime to \( n \), \( a^n = a^{(3)(2)} = a^{\phi(3^2)} \equiv 1 \pmod{3^2} \), and since \( 2 \) satisfies for \( n \), \( 2^{n^2} \mid a^n - 1 \), so this works.

Otherwise, let \( p_2 > p_1 = 3 \) be the least prime such that \( p_2 \mid n_1 \). From primitive roots, \( p_2(p_2 - 1) = \phi(p_2^2) \mid 2n_1 \), so by the definition of \( p_2 \), \( p_2 - 1 \mid 2p_2 = 2(3) \), so \( p_2 = 7 \).

If \( n = (2)(3)(7) \), then from Euler’s Theorem, for any positive integer \( a \) relatively prime to \( n \), \( a^n = a^{(7)(6)} = a^{\phi(7^2)} \equiv 1 \pmod{7^2} \), and since \( (2)(3) \) satisfies the condition for \( n \), \( 2^{n^2} \mid a^n - 1 \), so this works.

Otherwise, let \( p_3 > p_2 = 7 \) be the least prime such that \( p_3 \mid n_1 \). From primitive roots, \( p_3(p_3 - 1) = \phi(p_3^2) \mid 2n_1 \), so by the definition of \( p_3 \), \( p_3 - 1 \mid 2p_3p_2 = 2(3)(7) \), so \( p_3 = 43 \).

If \( n = (2)(3)(7)(43) \), then from Euler’s Theorem, for any positive integer \( a \) relatively prime to \( n \), \( a^n = a^{(43)(42)} = a^{\phi(7^2)} \equiv 1 \pmod{7^2} \), and since \( (2)(3)(7) \) satisfies the condition for \( n \), \( 2^{n^2} \mid a^n - 1 \), so this works.

If there exists a least prime \( p_4 > p_3 = 43 \) such that \( p_4 \mid n_1 \), then from primitive roots \( p_4(p_4 - 1) = \phi(p_4^2) \mid 2n_1 \), so by the definition of \( p_4 \),

\[
p_4 - 1 \mid 2p_1p_2p_3 = 2(3)(7)(43),
\]

so \( p_4 \in \{44, 87, 130, 259, 302, 603, 904, 1807\} \), so \( p_4 \) is not prime which is a contradiction.

Therefore all possible \( n \) are

\[
n \in \{2, (2)(3), (2)(3)(7), (2)(3)(7)(43)\} = \{2, 6, 42, 1806\}.
\]

**OC289.** Let \( a, b, c, d, e \) be distinct positive integers such that \( a^4 + b^4 = c^4 + d^4 = e^5 \). Show that \( ac + bd \) is a composite number.

*Originally problem 5 from day 2 of the 2015 USAMO.*

*No submitted solutions.*

**OC290.** Let \( \triangle ABC \) be a scalene triangle and \( X, Y \) and \( Z \) be points on the lines \( BC \), \( AC \) and \( AB \), respectively, such that \( \angle AXB = \angle BYC = \angle CZA \). The circumcircles of \( BXZ \) and \( CXY \) intersect at \( P \). Prove that \( P \) is on the circle whose diameter has ends in the orthocenter \( H \) and in the barycenter \( G \) of \( \triangle ABC \).

*Originally problem 6 from day 2 of the 2015 Brazil National Olympiad.*

*We present the solution by Andrea Fanchini. There were no other submissions.*

We use barycentric coordinates and the usual Conway’s notations with reference to triangle \( ABC \).
Then the generic points \(X, Y, Z\) have absolute coordinates
\[
X(0,v,1-v), \quad Y(1-w,0,w), \quad Z(u,1-u,0)
\]
where \(u, v\) and \(w\) are parameters.

Equation of line \(AX\) is \((v-1)y + vz = 0\), then the \(\angle AXB\) gives
\[
S_{AXB} = S \cot AXB = S_C - a^2v
\]
Equation of line \(BY\) is \(wx + (w-1)z = 0\), then the \(\angle BYC\) gives
\[
S_{BYC} = S \cot BYC = S_A - b^2w
\]
Equation of line \(CZ\) is \((u-1)x + uy = 0\), then the \(\angle CZA\) gives
\[
S_{CZA} = S \cot CZA = S_B - c^2u
\]
Now if \(\angle AXB = \angle BYC = \angle CZA\) we have the system
\[
\begin{align*}
S_C - a^2v &= S_B - c^2u \\
S_A - b^2w &= S_B - c^2u
\end{align*}
\]
Therefore points \(X\) and \(Y\) have coordinates that depend only on the parameter \(u\):
\[
X \left(0, \frac{b^2 - c^2 + c^2u}{a^2}, 2S_B - \frac{c^2u}{a^2}\right), \quad Y \left(\frac{a^2 - c^2u}{b^2}, 0, \frac{b^2 - a^2 + c^2u}{b^2}\right)
\]
Equation of a generic circle is
\[
a^2yz + b^2zx + c^2xy - (x + y + z)(px + qy + rz) = 0.
\]
If this circle passes through \(B, X, Z\) we obtain the three conditions
\[
q = 0, \quad r = b^2 - c^2 + c^2u, \quad p = c^2(1 - u),
\]
so this circle has equation
\[
a^2yz + b^2zx + c^2xy - (x + y + z) (c^2(1 - u)x + (b^2 - c^2 + c^2u)z) = 0
\]
and then the circumcircle \(CXY\) has equation
\[
a^2yz + b^2zx + c^2xy - (x + y + z) ( (b^2 - a^2 + c^2u)x + (2S_B - c^2u)y ) = 0
\]
so the radical axis is the line \(r\)
\[
r : 2(S_B - c^2u)x + (c^2u - 2S_B)y + (b^2 - c^2 + c^2u)z = 0.
\]

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Point $P$ (and $X$) is the intersection between the circle $CXY$ and the radical axis
\[
\begin{aligned}
&b^2x^2 + a^2y^2 + 2S_Cxy + (c^2u - a^2)x + (c^2 - b^2 - c^2u)y = 0, \\
y = \frac{(3S_B - S_C - 3c^2u)x + S_C - S_B + c^2u}{a^2}.
\end{aligned}
\]
Solving it we obtain the coordinates of $P$:
\[
\begin{aligned}
x - \text{coord} : 4S_B^2 + uc^2(S_C - 7S_B + 3uc^2) \\
y - \text{coord} : a^2c^2 - 2S_A S_B + 2S_B^2 + uc^2(S_A - 6S_B - S_C + 3uc^2) \\
z - \text{coord} : 2c^2S_B + uc^2(3uc^2 - 5S_B - S_A).
\end{aligned}
\]
This circle has as center the midpoint between $H(S_B S_C : S_C S_A : S_A S_B)$ and $G(1 : 1 : 1)$, that is
\[
M_{HG}(3S_B S_C + S^2 : 3S_C S_A + S^2 : 3S_A S_B + S^2).
\]
The radius $\rho$ is the distance $GM_{HG}$
\[
\rho^2 = \frac{S^2(S_A + S_B + S_C) - 9S_A S_B S_C}{36S^2},
\]
so the circle with diameter $HG$ has equation
\[
a^2yz + b^2zx + c^2xy - \frac{2}{3}(x + y + z)(S_A x + S_B y + S_C z) = 0.
\]
Now we put the coordinates of $P$ in this last equation and with a bit of algebra we can verify that this point is on the circle.