OC286. There are four basketball players $A, B, C, D$. Initially the ball is with $A$. The ball is always passed from one person to a different person. In how many ways can the ball come back to $A$ after seven moves? (For example, $A$ passes to $C$ who passes to $B$ who passes to $D$ who passes to $A$ who passes to $B$ who passes to $C$ who passes to $A$.)

Originally problem 4 of the 2015 India National Olympiad.

We received 4 correct solutions and 1 incorrect submission. We present the solution by Gabriel Wallace.

We claim there are 546 ways. First, we examine the range of $A$. It is trivial to show that the minimum number of times person $A$ has the ball is 2. Since no one can pass the ball to themselves, there has to be a space between each person, so the max number of times for $A$ is 4. Then we have three cases.

Case 1: $A = 4$. Here we have three possibilities, as follows:

$$
A - - A - A - A \\
A - A - - A - A
$$

A blank space directly following an $A$ can take 3 values: $B, C,$ or $D$. A space after that would have two possibilities, whatever two letters that were not selected prior. So a 1-blank will have 3 possibilities and a 2-blank will have 6. Notice in all 3 permutations, there are two 1-blanks and one 2-blank. Summing everything, we have $3(3 \cdot 3 \cdot 3) = 162$ ways for 4 $A$’s.

Case 2: $A = 3$. Here we have four possibilities, as follows:

$$
A - A - - - A \\
A - - A - - A \\
A - - - A - A \\
A - - - - A - A
$$

Directly following an $A$ we still have 3 possible values. Since we have already fixed the $A$’s, then any other space will have two possible values, whatever was not selected directly before. So here we have two lines with one 1-blank and one 4-blank, and two lines with one 2-blank and one 3 blank. Thus we have $2(3 \cdot 3 \cdot 2) + 2(3 \cdot 2 \cdot 3 \cdot 2^2) = 288$ ways for 3 $A$’s.

Case 3: $A = 2$. Here we have one possibility, as follows:

$$
A - - - - - A
$$
Following the same logic as above, for this one 6-blank we have $3 \cdot 2^5 = 96$ ways for 2 A’s.

Summing all the cases we have a total of $162 + 288 + 96 = 546$ ways.

**OC287.** Let

$$P(x) = ax^3 + (b - a)x^2 - (c + b)x + c$$

and

$$Q(x) = x^4 + (b - 1)x^3 + (a - b)x^2 - (c = a)x + c$$

be polynomials of $x$ with $a, b, c$ non-zero real numbers and $b > 0$. If $P(x)$ has three distinct real roots $x_0, x_1, x_2$ which are also roots of $Q(x)$, then:

1. Prove that $abc > 28$,
2. If $a, b, c$ are non-zero integers with $b > 0$, find all their possible values.

*Originally problem 2 of the 2015 Greece National Olympiad.*

We received 4 correct submissions. We present the solution by Ali Adnan.

Clearly the fourth root of $Q(x)$ must also be real. Let that root be $x_3$. Then from Viete’s relations,

$$x_0x_1x_2 = -\frac{c}{a}, \quad x_0x_1x_2x_3 = c,$$

$$x_0 + x_1 + x_2 = 1 - \frac{b}{a}, \quad x_0 + x_1 + x_2 + x_3 = 1 - b.$$

From the first set of equations, $x_3 = -a$ and using this in the second set, we get $\frac{b}{a} = b - a$. Again from Viete’s relation

$$x_0x_1 + x_1x_2 + x_2x_0 = -\frac{b + c}{a},$$

$$x_3(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_2x_0 = a - b.$$

The above two relations imply that

$$(-a) \cdot \left(1 - \frac{b}{a}\right) + -\frac{(b + c)}{a} = a - b \Rightarrow \frac{b}{a} + \frac{c}{a} = 2(b - a) \Rightarrow c = b.$$

So $abc = ab^2$, while

$$\frac{b}{a} = b - a \Rightarrow b = \frac{a^2}{a - 1}.$$

(Note that the above relation implies that $a > 1$).

Thus $ab^2 = a^3(a - 1)^{-2}$ ($= f(a)$, say). It is routine calculus to obtain $f'(a) = 0 \Rightarrow a = 5/3$ and check that $a = 5/3$ gives a minimum indeed for $f(a)$, $a > 1$. After that it is easily obtained that $f(5/3) > 28$, which proves part 1.
For part 2, it is seen that $b = \frac{a^2}{a-1}$ implies that

$$(a-1)|a^2 \Rightarrow (a-1)|(a^2 - (a^2 - 1)) \Rightarrow a - 1 = \pm 1.$$ 

But we also see that $a > 1$ and so $a - 1 = 1 \Rightarrow a = 2$. Thus $b = c = 4$. So in summary the only possible triplet of non-zero integral values of $(a,b,c)$ with $b > 0$ is $(2,4,4)$.

**OC288.** Find all positive integers $n$ such that for any positive integer $a$ relatively prime to $n$, $2n^2 \mid a^n - 1$.

*Originally problem 6 from day 2 of the 2015 Turkey National Olympiad.*

We present the solution by Steven Chow. There were no other submissions.

If $n \equiv 1 \pmod{2}$, then since there are infinitely many primes, there exists integer $a$ relatively prime to $n$ such that $a \equiv 0 \pmod{2}$, so $2n^2 \nmid a^n - 1$, which is a contradiction.

Let $n_1$ be the integer such that $2n_1 = n$. Therefore

$$2n^2 \mid a^n - 1 \Rightarrow 2(2n_1)^2 \mid a^{2n_1} - 1 \Rightarrow 2^3 n_1^2 \mid (a^{n_1} + 1)(a^{n_1} - 1).$$

Let $2^k \mid n_1$. From Dirichlet’s Theorem, there exists positive integer $a$ relatively prime to $n = 2n_1$ such that $a \equiv 5 \pmod{8}$, so

$$2^3 n_1^2 \mid (a^{n_1} + 1)(a^{n_1} - 1) = \left(\prod_{j=0}^{k} \left(2^{2^{j}}a^{2^{j}} + 1\right)\right)\left(2^{2^{k}}a^{2^{k}} - 1\right)$$

$$\Rightarrow 3 + 2k \leq (k+1) + 2 \Rightarrow k \leq 0.$$ 

Therefore $k = 0$.

If $n_1$ is not divisible by a prime, then $n_1 = 1$ and $n = 2$. For all positive integers $a$ relatively prime to $2 = n$, $2n^2 = 2(2)^2 \mid a^2 - 1 = a^n - 1$, so this works.

Otherwise, $n_1$ is divisible by a prime.

If there exists a prime $p \geq 3$ and an integer $m$ such that $p^m \mid n_1$, then

$$p^{2m} \mid n_1^2 \mid a^{2n_1} - 1 \Rightarrow a^{2n_1} \equiv 1 \pmod{p^{2m}},$$

so since $p^{2m}$ has primitive roots (well known), we have

$$p^{2m-1} \mid p^{2m-1}(p-1) = \phi(p^{2m}) \mid 2n_1 \Rightarrow 2m - 1 \leq m \Rightarrow m \leq 1.$$ 

Therefore $n$ is square free.

Let $p_1$ be the least prime such that $p_1 \mid n_1$. Since $k = 0$, $p_1 \geq 3$. By definition, $p_1 - 1 \nmid n_1$, and from above, $p_1(p_1 - 1) \mid 2n_1$, so $p_1 = 3$.

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If $n = (2)(3)$, then from Euler’s Theorem, for any positive integer $a$ relatively prime to $n$, $a^n = a^{(3)(2)} = a^{\phi(3^2)} \equiv 1 \pmod{3^2}$, and since 2 satisfies for $n$, $2n^2 | a^n - 1$, so this works.

Otherwise, let $p_2 > p_1 = 3$ be the least prime such that $p_2 | n_1$. From primitive roots, $p_2(p_2 - 1) = \phi(p_2^2) | 2n_1$, so by the definition of $p_2$, $p_2 - 1 | 2p_2 = 2(3)$, so $p_2 = 7$.

If $n = (2)(3)(7)$, then from Euler’s Theorem, for any positive integer $a$ relatively prime to $n$, $a^n = a^{(7)(6)} = a^{\phi(7^2)} \equiv 1 \pmod{7^2}$, and since $(2)(3)$ satisfies the condition for $n$, $2n^2 | a^n - 1$, so this works.

Otherwise, let $p_3 > p_2 = 7$ be the least prime such that $p_3 | n_1$. From primitive roots, $p_3(p_3 - 1) = \phi(p_3^2) | 2n_1$, so by the definition of $p_3$, $p_3 - 1 | 2p_1p_2 = 2(3)(7)$, so $p_3 = 43$.

If $n = (2)(3)(7)(43)$, then from Euler’s Theorem, for any positive integer $a$ relatively prime to $n$, $a^n = a^{(43)(42)} = a^{\phi(7^2)} \equiv 1 \pmod{7^2}$, and since $(2)(3)(7)$ satisfies the condition for $n$, $2n^2 | a^n - 1$, so this works.

If there exists a least prime $p_4 > p_3 = 43$ such that $p_4 | n_1$, then from primitive roots $p_4(p_4 - 1) = \phi(p_4^2) | 2n_1$, so by the definition of $p_4$,

$$p_4 - 1 | 2p_1p_2p_3 = 2(3)(7)(43),$$

so $p_4 \in \{44, 87, 130, 259, 302, 603, 904, 1807\}$, so $p_4$ is not prime which is a contradiction.

Therefore all possible $n$ are

$$n \in \{2, (2)(3), (2)(3)(7), (2)(3)(7)(43)\} = \{2, 6, 42, 1806\}.$$  

**OC289.** Let $a, b, c, d, e$ be distinct positive integers such that $a^4 + b^4 = c^4 + d^4 = e^5$. Show that $ac + bd$ is a composite number.

*Originally problem 5 from day 2 of the 2015 USAMO.*

*No submitted solutions.*

**OC290.** Let $\triangle ABC$ be a scalene triangle and $X, Y$ and $Z$ be points on the lines $BC, AC$ and $AB$, respectively, such that $\angle AXB = \angle BYC = \angle CZA$. The circumcircles of $BXZ$ and $CXY$ intersect at $P$. Prove that $P$ is on the circle whose diameter has ends in the orthocenter $H$ and in the barycenter $G$ of $\triangle ABC$.

*Originally problem 6 from day 2 of the 2015 Brazil National Olympiad.*

*We present the solution by Andrea Fanchini. There were no other submissions.*

We use barycentric coordinates and the usual Conway’s notations with reference to triangle $ABC$. 

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Then the generic points $X, Y$ and $Z$ have absolute coordinates

$$X(0, v, 1 - v), \quad Y(1 - w, 0, w), \quad Z(u, 1 - u, 0)$$

where $u, v$ and $w$ are parameters.

Equation of line $AX$ is $(v - 1)y + vz = 0$, then the $\angle AXB$ gives

$$S_{AXB} = S \cot AXB = S_C - a^2 v$$

Equation of line $BY$ is $wx + (w - 1)z = 0$, then the $\angle BYC$ gives

$$S_{BYC} = S \cot BYC = S_A - b^2 w$$

Equation of line $CZ$ is $(u - 1)x + uy = 0$, then the $\angle CZA$ gives

$$S_{CZA} = S \cot CZA = S_B - c^2 u$$

Now if $\angle AXB = \angle BYC = \angle CZA$ we have the system

$$\begin{cases}
S_C - a^2 v = S_B - c^2 u \\
S_A - b^2 w = S_B - c^2 u
\end{cases} \Rightarrow \begin{cases}
v = \frac{b^2 - a^2 + c^2 u}{a^2} \\
w = \frac{b^2 - a^2 + c^2 u}{b^2}
\end{cases}$$

Therefore points $X$ and $Y$ have coordinates that depend only on the parameter $u$:

$$X \left( 0, \frac{b^2 - c^2 + c^2 u}{a^2}, 2S_B - c^2 u \right), \quad Y \left( \frac{a^2 - c^2 u}{b^2}, 0, \frac{b^2 - a^2 + c^2 u}{b^2} \right).$$

Equation of a generic circle is

$$a^2 yz + b^2 zx + c^2 xy - (x + y + z)(px + qy + rz) = 0.$$  

If this circle passes through $B, X, Z$ we obtain the three conditions

$$q = 0, \quad r = b^2 - c^2 + c^2 u, \quad p = c^2 (1 - u),$$

so this circle has equation

$$a^2 yz + b^2 zx + c^2 xy - (x + y + z) \left( c^2 (1 - u)x + (b^2 - c^2 + c^2 u)z \right) = 0$$

and then the circumcircle $CXY$ has equation

$$a^2 yz + b^2 zx + c^2 xy - (x + y + z) \left( (b^2 - a^2 + c^2 u)x + (2S_B - c^2 u)y \right) = 0$$

so the radical axis is the line $r$

$$r : 2(S_B - c^2 u)x + (c^2 u - 2S_B)y + (b^2 - c^2 + c^2 u)z = 0.$$  

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Point $P$ (and $X$) is the intersection between the circle $CXY$ and the radical axis

\[
\begin{align*}
\begin{cases}
b^2x^2 + a^2y^2 + 2S_Cxy + (c^2u - a^2)x + (c^2 - b^2 - c^2u)y = 0, \\
y = \frac{(3S_B - S_C - 3c^2u)x + S_C - S_B + c^2u}{a^2}.
\end{cases}
\end{align*}
\]

Solving it we obtain the coordinates of $P$:

\[
\begin{align*}
&x - \text{coord} : 4S_B^2 + uc^2(S_C - 7S_B + 3uc^2) \\
y - \text{coord} : a^2c^2 - 2S_AS_B + 2S_B^2 + uc^2(S_A - 6S_B - S_C + 3uc^2) \\
z - \text{coord} : 2c^2S_B + uc^2(3uc^2 - 5S_B - S_A).
\end{align*}
\]

This circle has as center the midpoint between $H(S_BS_C : S_CSA : S_AS_B)$ and $G(1 : 1 : 1)$, that is

\[M_{HG}(3S_BS_C + S^2 : 3S_CSA + S^2 : 3S_AS_B + S^2).\]

The radius $\rho$ is the distance $GM_{HG}$

\[
\rho^2 = \frac{S^2(S_A + S_B + S_C) - 9S_AS_BS_C}{36S^2},
\]

so the circle with diameter $HG$ has equation

\[
a^2yz + b^2zx + c^2xy - \frac{2}{3}(x + y + z)(S_Ax + S_By + S_Cz) = 0.
\]

Now we put the coordinates of $P$ in this last equation and with a bit of algebra we can verify that this point is on the circle.