FOCUS ON...
No. 27
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Some relations in the triangle (I)

Introduction

Relations between the elements of a triangle intervene in *Crux* problems – and solutions – quite often, and that's an understatement! Every solver wanting to establish an identity or an inequality involving those elements should have a number of these relations in her/his toolbox: the Laws of Sines and Cosines of course, but also classical relations such as $a = b \cos C + c \cos B$ or $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.

However, even the most common of these relations form a vast subject; for instance, the compendium of them in [1] spreads over some twenty pages! The modest goal of this number and the next one is to present a selection of less familiar relations chosen because of their aesthetic qualities and/or their applications to problems. Here and in what follows, we use the standard notations as they can be found in [2] (where $\alpha, \beta, \gamma$ are preferred to $A, B, C$ for the angles of the triangle, though).

About the differences of angles $B - C, C - A$ and $A - B$

We begin with the surprising relation

$$a \cos(B - C) + b \cos(C - A) + c \cos(A - B) = 2(a \cos A + b \cos B + c \cos C). \quad (1)$$

The proof is easy: we have that $a \sin C = c \sin A$ (from the Law of Sines) and $b = a \cos C + c \cos A$, therefore

$$a \cos(B - C) - b \cos B = a \cos B \cos C + a \sin B \sin C - b \cos B$$
$$= \cos B(a \cos C - b) + c \sin A \sin B$$
$$= -c \cos A \cos B + c \sin A \sin B$$
$$= -c \cos(A + B)$$
$$= c \cos C.$$

Hence

$$a \cos(B - C) = b \cos B + c \cos C.$$

With similar results for $b \cos(C - A)$ and $c \cos(A - B)$, the desired equality is deduced at once.

With the help of the Law of Cosines, (1) leads to

$$a \cos(B - C) + b \cos(C - A) + c \cos(A - B)$$
$$= \frac{2(a^2b^2 + b^2c^2 + c^2a^2 - a^4 - b^4 - c^4)}{abc} \quad (2)$$
so that
\[ a \cos(B - C) + b \cos(C - A) + c \cos(A - B) = \frac{16F^2}{4RF} = \frac{4F}{R} = \frac{abc}{R^2} = \frac{4rs}{R}. \]

Here are two applications. First, the latter shows that \( \frac{4rs}{R} \leq a + b + c = 2s \) and we obtain Euler's inequality \( R \geq 2r \) in a rather oblique way!

Second, (1) yields a neat solution to problem 2546 [2000 : 237 ; 2001 : 343]

Prove that triangle \( ABC \) is equilateral if and only if
\[ a \cos(B - C) + b \cos(C - A) + c \cos(A - B) = \frac{a^4 + b^4 + c^4}{abc}, \]
for (2) shows that the given relation is equivalent to
\[ 2(a^4 + b^4 + c^4) = 2(a^2b^2 + b^2c^2 + c^2a^2), \]
that is, to
\[ (a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 = 0, \]
which clearly holds if and only if \( a = b = c \).

In the same vein, we will prove the following beautiful formula
\[ a^3 \cos(B - C) + b^3 \cos(C - A) + c^3 \cos(A - B) = 3abc. \] (3)

To this aim, we start with expressions of the area \( F \) that deserve to be better known:
\[ 4F = a^2 \sin 2B + b^2 \sin 2A = b^2 \sin 2C + c^2 \sin 2B = c^2 \sin 2A + a^2 \sin 2C. \] (4)

This follows, for example, from
\[ a^2 \sin 2B + b^2 \sin 2A = 2a^2 \sin B \cos B + 2ab \sin B \cos A = 2ac \sin B = 4F. \]

Formula (3) is deduced smoothly once we have noticed that
\[ a^3 \cos(B - C) = a^2 \cdot 2R \sin A \cos(B - C) \]
\[ = 2Ra^2 \sin (B + C) \cos(B - C) \]
\[ = Ra^2 (\sin 2B + \sin 2C). \]

Then, using (4),
\[ \sum_{\text{cyclic}} a^3 \cos(B - C) \]
\[ = R(a^2 \sin 2B + a^2 \sin 2C + b^2 \sin 2C + b^2 \sin 2A + c^2 \sin 2A + c^2 \sin 2B) \]
\[ = R(4F + 4F + 4F) \]
\[ = 12RF \]
\[ = 3abc. \]
About the cosines of $\frac{B-C}{2}$, $\frac{C-A}{2}$ and $\frac{A-B}{2}$

We will consider two expressions of $\cos\left(\frac{B-C}{2}\right)$ and apply them to past problems. The first one is readily obtained:

$$\cos\left(\frac{B-C}{2}\right) = \frac{b+c}{a} \cdot \sin \frac{A}{2}. \quad (5)$$

Indeed, because $\cos \frac{A}{2} = \sin \left(\frac{B+C}{2}\right)$, we have

$$\cos\left(\frac{B-C}{2}\right) \sin \frac{A}{2} = \frac{2 \sin \left(\frac{B+C}{2}\right) \cos \left(\frac{B-C}{2}\right)}{2 \cos \frac{A}{2} \sin \frac{A}{2}} = \frac{\sin B + \sin C}{\sin A} = \frac{b+c}{a}.$$

Relation (5) and similar relations provide a quick and easy solution to problem [2002 : 112; 2003 : 119], which states:

For any triangle $ABC$, prove that

$$8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \leq \cos \left(\frac{A-B}{2}\right) \cos \left(\frac{B-C}{2}\right) \cos \left(\frac{C-A}{2}\right).$$

This immediately follows from (5):

$$\frac{\cos \left(\frac{B-C}{2}\right) \cos \left(\frac{C-A}{2}\right) \cos \left(\frac{A-B}{2}\right)}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{\left(\frac{b+c}{a}\right) \left(\frac{c+a}{b}\right) \left(\frac{a+b}{c}\right)}{2 + \left(\frac{b}{a} + \frac{a}{b}\right) + \left(\frac{c}{b} + \frac{b}{c}\right) + \left(\frac{a}{c} + \frac{c}{a}\right)} \geq 8,$$

since $x + \frac{1}{x} \geq 2$ for all positive real numbers $x$.

Note that the inequality rewrites as

$$\cos \left(\frac{A-B}{2}\right) \cos \left(\frac{B-C}{2}\right) \cos \left(\frac{C-A}{2}\right) \geq 2r. \quad (6)$$

From (6) we easily derive the related inequality

$$\cos \left(\frac{A-B}{2}\right) + \cos \left(\frac{B-C}{2}\right) + \cos \left(\frac{C-A}{2}\right) \geq 1 + \frac{4r}{R}. \quad (7)$$

Proof. Since

$$\cos \left(\frac{A-B}{2}\right) \geq \cos^2 \left(\frac{A-B}{2}\right) = \frac{1}{2} + \frac{1}{2} \cos(A-B)$$

it is sufficient to show that

$$\cos(A-B) + \cos(B-C) + \cos(C-A) \geq \frac{8r}{R} - 1.$$
But, when \( x + y + z = 0 \), we have
\[
\cos x + \cos y + \cos z = 4 \cos \frac{x}{2} \cos \frac{y}{2} \cos \frac{z}{2} - 1,
\]
hence the latter inequality readily transforms into (6) and therefore is true.

It is interesting to notice that this is a variant of proof of an old \textit{Crux} inequality, namely
\[
\cos \left( \frac{A - B}{2} \right) + \cos \left( \frac{B - C}{2} \right) + \cos \left( \frac{C - A}{2} \right) \geq 4(\cos A + \cos B + \cos C) - 3,
\]
proposed in problem \textbf{696} \cite{1981:302;1982:316} (taking into account the known formula \( \cos A + \cos B + \cos C = 1 + \frac{r}{R} \)).

The following additional formula involves the distance \( IA \) from the incenter \( I \) to the vertex \( A \):
\[
\cos \left( \frac{B - C}{2} \right) = \frac{IA}{2R} + \frac{r}{IA}.
\]

To prove this relation, we introduce the circumcentre \( O \) of \( ABC \) and observe that in the case when \( B \geq C \), we have \( C \leq 90^\circ \) and
\[
\angle BAO = \angle OBA = \frac{180^\circ - \angle AOB}{2} = 90^\circ - C.
\]

In consequence,
\[
\angle IAO = \angle BAO - \angle BAI = 90^\circ - C - \frac{A}{2} = \frac{B - C}{2}.
\]

If \( C > B \), we obtain \( \angle IAO = \frac{C - B}{2} \) and in either case
\[
\cos \left( \frac{B - C}{2} \right) = \frac{IA^2 + R^2 - IO^2}{2IA \cdot R} = \frac{IA}{2R} + \frac{r}{IA}
\]
since \( IO^2 = R^2 - 2rR \) (Euler’s formula).

Note that an application of the arithmetic-geometric mean inequality yields
\[
\cos \left( \frac{B - C}{2} \right) \geq 2\sqrt{\frac{IA}{2R} \cdot \frac{r}{IA}} = 2\sqrt{\frac{r}{2R}}
\]
or

\[ \cos^2 \left( \frac{B - C}{2} \right) \geq \frac{2r}{R}, \]

the inequality to be proved in problem 2382 [1998 : 425 ; 1999 : 440].

Our second part will offer relations involving the altitudes, exradii, etc. By way of transition, let us remark that if \( H \) is the orthocentre of \( \Delta ABC \), then \( \angle IAO = \angle IAH \) (recall that the line \( AO \) and the altitude from vertex A are symmetric in the angle bisector of \( \angle BAC \)). We deduce that \( \cos \left( \frac{B - C}{2} \right) = \frac{h_a}{w_a} \) where, following the notations of [2], \( w_a = AD \) is the length of the angle bisector of \( \angle BAC \) (see figure above). Therefore, relations (6) and (7) yield

\[
\frac{h_a h_b h_c}{w_a w_b w_c} \geq \frac{2r}{R} \quad \text{and} \quad \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \geq 1 + \frac{4r}{R}.
\]

**Exercises**

1. a) Establish the formula \( \sin 2A + \sin 2B + \sin 2C = \frac{abc}{2R^3} \) and deduce an expression of \( a^2 \sin 2A + b^2 \sin 2B + c^2 \sin 2C \).

b) Prove that

\[
a^3 \cos A + b^3 \cos B + c^3 \cos C = \frac{abc}{2R^2} \cdot (a^2 + b^2 + c^2 - 6R^2)
\]

and

\[
a \cos^3 A + b \cos^3 B + c \cos^3 C = \frac{abc}{8R^4} \cdot (10R^2 - (a^2 + b^2 + c^2)).
\]

c) From the latter, deduce that if \( \Delta ABC \) is not obtuse then

\[ a \cos^3 A + b \cos^3 B + c \cos^3 C \leq \frac{abc}{4R^2}, \]

(the inequality to be proved in problem 3167 [2006 : 395,397 ; 2007 : 374].)

2. Prove the inequality \( \sum_{\text{cyclic}} a \cos \frac{B-C}{2} \geq s \left(1 + \frac{2r}{R}\right) \) (use (5) or give a look at problem 696) and deduce that

\[ \frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \geq \frac{1}{2r} + \frac{1}{R}. \]

**References**


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