CONTEST CORNER SOLUTIONS


CC226. In the table below we write all the different products of two distinct counting numbers between 1 and 100:

\[
\begin{array}{cccc}
1 \cdot 2, & 1 \cdot 3 & \ldots & 1 \cdot 99, & 1 \cdot 100 \\
2 \cdot 3 & \ldots & 2 \cdot 99 & 2 \cdot 100 \\
\vdots & \vdots & \vdots & \vdots \\
99 \cdot 100 \\
\end{array}
\]

Find the sum of all of these products.

Originally problem 46 from the Thirty-fifth Annual Columbus State Invitational Mathematics Tournament (2009).

We received 17 solutions of which 16 were correct and complete. We present 2 solutions.

Solution 1, by Miguel Amengual Covas.

We add all entries of the given table by adding the entries of each column and then adding the column sums. We obtain

\[
1 \cdot 2 + (1 + 2) \cdot 3 + \ldots + (1 + 2 + \ldots + 98) \cdot 99 + (1 + 2 + \ldots + 99) \cdot 100.
\]

Since each parenthesis consists of a sum of consecutive integers starting with 1, this expression may be rewritten as

\[
1 \cdot 2 + \frac{2(1 + 2)}{2} \cdot 3 + \ldots + \frac{98(1 + 98)}{2} \cdot 99 + \frac{99(1 + 99)}{2} \cdot 100
\]

\[
= \frac{1}{2} \left( 1 \cdot 2^2 + 2 \cdot 3^2 + \ldots + 98 \cdot 99^2 + 99 \cdot 100^2 \right).
\]

Now we use the summation notation and write the last expression as

\[
\frac{1}{2} \sum_{i=1}^{99} i(i + 1)^2,
\]

obtaining

\[
\frac{1}{2} \sum_{i=1}^{99} i(i + 1)^2 = \frac{1}{2} \sum_{i=1}^{99} \left( i^3 + 2i^2 + i \right) = \frac{1}{2} \left( \sum_{i=1}^{99} i^3 + 2 \sum_{i=1}^{99} i^2 + \sum_{i=1}^{99} i \right). \quad (1)
\]

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Applying the formulas
\[
\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2, \\
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \\
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\]
with \(n = 99\), we get
\[
\sum_{i=1}^{99} i^3 = 24502500, \quad \sum_{i=1}^{99} i^2 = 656700, \quad \sum_{i=1}^{99} i = 4950.
\]
When these results are substituted into (1), we get
\[
\frac{1}{2} \sum_{i=1}^{99} i(i+1)^2 = 12582075.
\]

**Solution 2, by Ivko Dimitrić.**

A suggestive upper triangular display of the products \(i \cdot j\) in the \(i\)th row and \(j\)th column (\(i < j\)) prompts us to consider also the missing part of the 100 \(\times\) 100 symmetric square array, filling in the lower triangular part with products for which \(i > j\) and diagonal with products \(i \cdot i\) for \(1 \leq i, j \leq 100\). Because of the symmetry of the completed array about the main diagonal (\(i \cdot j = j \cdot i\)), the sum we are asked to find also appears as the sum of all the entries below the diagonal. Therefore, we find that sum by subtracting the sum of the diagonal entries from the sum of all the entries in the table and dividing the rest by two. The sum of all the entries in the square table equals
\[
\sum_{j=1}^{100} \sum_{i=1}^{100} i \cdot j = \left(\sum_{i=1}^{100} i\right) \left(\sum_{j=1}^{100} j\right) = \left(\sum_{i=1}^{100} i\right)^2 = \left(\frac{100 \cdot 101}{2}\right)^2,
\]
and we need also to recall the well-known summation formulas
\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.
\]
Then the sum we are asked to compute equals
\[
\frac{1}{2} \left[ \left(\sum_{i=1}^{100} i\right)^2 - \sum_{i=1}^{100} i^2 \right] = \frac{1}{2} \left[ \left(\frac{100 \cdot 101}{2}\right)^2 - \frac{100 \cdot 101 \cdot 201}{6} \right] = 12582075.
\]
CC227. Suppose \( \{a_1, a_2, \ldots \} \) is a geometric sequence of real numbers. The sum of the first \( n \) terms is \( S_n \). If \( S_{10} = 10 \) and \( S_{30} = 70 \), determine the value of \( S_{40} \).

Adapted from problem 36 from the Forty-second Annual Columbus State Invitational Mathematics Tournament (2016).

We received 14 correct solutions and one incorrect submission. We present the solution of Ivko Dimitrić.

Let \( r \) be the common ratio of the sequence. Then the \( n^{th} \) term is \( a_n = a_1 r^{n-1} \) and the sum of the first \( n \) terms is

\[
S_n = \frac{a_1 (1 - r^n)}{1 - r}.
\]

(Clearly, from the information given, \( r \neq 1 \)). We are given

\[
S_{10} = \frac{a_1 (1 - r^{10})}{1 - r} = 10 \quad \text{and} \quad S_{30} = \frac{a_1 (1 - r^{30})}{1 - r} = 70.
\]

Dividing the two we have,

\[
7 = \frac{S_{30}}{S_{10}} = \frac{1 - (r^{10})^3}{1 - r^{10}} = 1 + r^{10} + (r^{10})^2.
\]

Setting \( q = r^{10} \) we get a quadratic equation \( q^2 + q + 1 = 7 \) with roots \( q = -3 \) and \( q = 2 \). The first root must be discarded since it cannot equal an even power of a real number. Thus \( q = r^{10} = 2 \). Next we compute the ratio,

\[
\frac{S_{40}}{S_{10}} = \frac{1 - r^{40}}{1 - r^{10}} = (1 + r^{10})(1 + r^{20}) = (1 + 2)(1 + 2^2) = 15.
\]

Therefore \( S_{40} = 15 \cdot S_{10} = 150 \).

CC228. In the triangle \( ABC \), \( AB = 2\sqrt{13}, AC = \sqrt{73} \), \( E \) and \( D \) are the midpoints of \( AB \) and \( AC \), respectively. Furthermore, \( BD \) is perpendicular to \( CE \). Find the length of \( BC \).

Originally problem 48 from the Forty-second Annual Columbus State Invitational Mathematics Tournament (2016).

We received 12 correct solutions and two incorrect submissions; we present two solutions.
Solution 1, by David Manes.
Since \( G \) is the centroid of triangle \( ABC \) it splits each median into a 2:1 ratio. Let \( |EG| = x \) and \( |DG| = y \), which means \( |GC| = 2x \) and \( |GB| = 2y \). Applying the Pythagorean Theorem to each of the right triangles \( EGB \), \( CGD \) and \( CBG \) we get

\[
\begin{align*}
x^2 + 4y^2 &= (\sqrt{13})^2 \\
4x^2 + y^2 &= \left(\frac{\sqrt{73}}{2}\right)^2 \\
4x^2 + 4y^2 &= |BC|^2
\end{align*}
\]

Adding equations (1) and (2) and multiplying both sides of the result by \( \frac{4}{5} \) we get

\[4x^2 + 4y^2 = 25,\]

which combined with (3) allows us to conclude that \( |BC| = 5 \).

Solution 2, by Titu Zvonaru.
Draw in the segment \( ED \); since it joins two midpoints, \( |ED| = \frac{1}{2} |BC| \). The diagonals of quadrilateral \( BCDE \) are perpendicular, and hence the sum of the squares of opposite sides is constant; that is,

\[
|BC|^2 + |ED|^2 = |BE|^2 + |CD|^2 \iff
\]

\[
|BC|^2 + \frac{|BC|^2}{4} = \frac{52}{4} + \frac{73}{4} \iff
\]

\[5|BC|^2 = 125,\]

whence we obtain \( |BC| = 5 \).

CC229. A store has objects that cost either 10, 25, 50, or 70 cents. If Sharon buys 40 objects and spends seven dollars, what is the largest quantity of the 50 cent items that could have been purchased?

*Originally problem 37 from the Thirty-fifth Annual Columbus State Invitational Mathematics Tournament (2009).*

We received eight correct solutions. We present the solution of Hannes Geupel.

If 8 of the 40 objects cost each 50 cents, Sharon has to buy 32 objects that cost 10 cents to keep the price as small as possible. Altogether these 40 objects cost $7.20. So Sharon has to buy less than 8 items that cost 50 cents.

If 7 of the 40 objects cost each 50 cents, she has to buy 33 objects that cost 10 cents to keep the price as small as possible. These 40 objects cost $6.80 in total. So Sharon has to buy more expensive objects. The minimum price that can be added to this $6.80 is 15 cents, by buying one 10 cent object less and one 25 cent object instead. Then the objects cost $6.95. This is still not enough. So Sharon

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has to buy another more expensive object. The minimum price that can be added is 15 cents again, by buying one 10 cent object less and one 25 cent object instead. Then the 40 objects cost $7.10. So Sharon has to buy less than 7 items that cost 50 cents.

If 6 of the 40 objects cost each 50 cents, it is possible to spend $7 for all objects by buying 33 items that cost 10 cents, 0 that cost 25 cents, 6 that cost 50 cents, and 1 that costs 70 cents. Thus, 6 is the largest number of 50 cent items she could have bought.

**CC230.** Two friends agree to meet at the library between 1:00 P.M. and 2:00 P.M. Each agrees to wait 20 minutes for the other. What is the probability that they will meet if their arrivals occur at random during the hour and if the arrival times are independent?

*Originally problem 38 from the Thirty-fifth Annual Columbus State Invitational Mathematics Tournament (2009).*

*We received seven correct solutions. We present the solution of Ángel Plaza.*

Let us use as sample space the unit square. Let $X$ and $Y$ be the two independent, randomly chosen times by person $X$ and $Y$ respectively from the one hour period.

These assumptions imply that every point in the unit square is equally likely, where the first coordinate represents the time when person $X$ shows up and the second coordinate represents when person $Y$ shows up. As 20 minutes is one third of the time between 1 pm and 2 pm, the required probability in this situation is the area of the the set of all points lying in region $A$ in the figure below.

This can be seen as the unit square minus two triangles, giving the area of $A$ as

$$1 - 2 \left( \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3} \right) = 1 - \frac{4}{9} = \frac{5}{9}.$$