The \textit{pqr} Method: Part II

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The first part of this article discussed the \textit{pqr} Lemma. In this part, we will examine its extension. For completeness, we re-state the \textit{pqr} Lemma here:

\textbf{The \textit{pqr} Lemma.} For three complex numbers \(a, b\) and \(c\), let \(p = a + b + c\), \(q = ab + bc + ca\), \(r = abc\), and define

\[
T(p, q, r) = -4p^3r + p^2q^2 + 18pqr - 4q^3 - 27r^2 = (a - b)^2(b - c)^2(c - a)^2.
\]

When we fix two of \(p, q, r\) such that there exist triples \((p, q, r)\) satisfying \(p, q, r \geq 0\) and \(T(p, q, r) \geq 0\), the unfixed variable obtains its maximum and minimum values when two of \(a, b, c\) are equal. There is one exception – when \(r\) is the unfixed variable, its minimum value occurs when either two of \(a, b, c\) are equal, or one of them is equal to 0.

\section*{Special Conditions}

Consider the following problem:

Let \(a, b, c\) be non-negative numbers such that \(a^2 + b^2 + c^2 + abc = 4\). Prove that

\[
a + b + c \geq 2 + \sqrt{abc(4 - a - b - c)}.
\]

Here, the condition \(a^2 + b^2 + c^2 + abc = 4\) is equivalent to \(p^2 - 2q + r = 4\). However, fixing two of \(p, q, r\) would fix the third, so we cannot apply the \textit{pqr} lemma directly.

Fortunately, we can extend the \textit{pqr} method to deal with conditions like these.

\textbf{The \textit{pqr} Lemma For Special Conditions.} Let the variables \(a, b, c\) obey a symmetric condition that can be written as \(G(p, q, r) = 0\), where \(G\) is a continuous function. Let \((x, y, z)\) be a permutation of \((p, q, r)\). Fix some value \(z \geq 0\) such that there exist triples \((p, q, r)\) satisfying \(p, q, r \geq 0\), \(T(p, q, r) \geq 0\), and \(G(p, q, r) = 0\).

Assume that the condition \(G\) is equivalent to \(y = f(x)\) (while \(z\) is fixed), where a set of valid values of \(x\) is bounded, and \(f\) is continuous over that set. If \(z = r\), \(x\) obtains its maximum and minimum values when two of \(a, b, c\) are equal. If \(z \neq r\), \(x\) obtains its maximum and minimum values when either two of \(a, b, c\) are equal, or one of them equals 0.

We will prove this for \((x, y, z) = (q, p, r)\), because the proofs for the others are similar.

\textit{Proof.} When \(r\) is fixed, each of the inequalities \(T(f(q), q, r) = T(p, q, r) \geq 0\) and \(f(q) = p \geq 0\) defines a union of several intervals and rays. The set of valid...
values of \( q \) is the intersection of those sets, so it is a union of several intervals (it cannot contain rays because we are assuming that this set is bounded). At the endpoints of those intervals, either \( T(p, q, r) = 0 \) or \( p = 0 \). The maximum and minimum values of \( q \) must be at an endpoint. Therefore \( q \) attains its maximum and minimum values when two of \( a, b, c \) are equal. \( \square \)

Now, we can use this to solve the problem above.

Solution. The inequality
\[
a + b + c \geq 2 + \sqrt{abc(4 - a - b - c)}
\]
is equivalent to \( p \geq 2 + \sqrt{r(4 - p)} \), and the condition is equivalent to
\[
q = \frac{p^2 - 4 + r}{2}.
\]
If we fix \( r \), \( q \) is a continuous function of \( p \). Let \( f(p) = p - 2 - \sqrt{r(4 - p)} \). Then, the inequality can be written as \( f(p) \geq 0 \). Since \( f(p) \) is monotonic, it suffices to prove the inequality for the minimum value of \( p \). From the above lemma, this occurs when two of the variables are equal.

WLOG assume that \( a = b \). The condition becomes
\[
2a^2 + c^2 + a^2c = 4, \quad \text{or} \quad (c + 2)(c + a^2 - 2) = 0,
\]
so \( c = 2 - a^2 \). Substituting this into the inequality, we get \( a^4(a - 1)^2 \geq 0 \). Since it is true for the minimum value of \( p \), it is true for all values of \( p \), and we are done.

Below is an additional example to further show the usefulness of the \( pqr \) method:

Example. Let \( a, b, c \) be non-negative real numbers such that
\[
a^2 + b^2 + c^2 = ab + bc + ca + (abc - 1)^2.
\]
Prove that
\[
ab + bc + ca + 3 \geq 2(a + b + c).
\]

Solution. The condition is equivalent to \( q = \frac{p^2 - (r - 1)^2}{3} \). Plugging in this value of \( q \) into the desired inequality yields
\[
\frac{p^2 - (r - 1)^2}{3} + 3 - 2p \geq 0.
\]
When \( p \) is fixed, this is a concave function in terms of \( r \), and \( q \) is a continuous function of \( r \). Therefore, it is only necessary to consider the extreme values of \( r \). By the \( pqr \) lemma for special conditions, \( r \) takes an extreme value when WLOG \( a = 0 \) or \( a = b \).

If \( a = 0 \), then the condition is \( b^2 + c^2 = bc + 1 \), which is equivalent to
\[
(b + c)^2 = 3bc + 1.
\]
By the AM-GM inequality,

\[(b + c)^2 = 3bc + 1 \leq \frac{3(b + c)^2}{4} + 1.\]

From this inequality, \(b + c \leq 2\). The desired inequality is

\[bc + 3 \geq 2(b + c).\]

By the condition, \(bc = \frac{(b+c)^2-1}{3}\), so it is enough to prove that

\[\frac{(b + c)^2 - 1}{3} + 3 \geq 2(b + c).\]

This inequality is equivalent to \((2 - b - c)(4 - b - c) \geq 0\), which is true since \(b + c \leq 2\).

If \(a = b\), then the desired inequality is equivalent to \(a^2 + 2ac + 3 \geq 4a + 2c\). The condition becomes

\[(a - c)^2 = (a^2c - 1)^2,\]

which means that \(a - c = a^2c - 1\) or \(a - c = 1 - a^2c\). If \(a - c = a^2c - 1\), then

\[c = \frac{a + 1}{a^2 + 1}.\]

Plugging this into the inequality yields

\[a^2 + 2a \left( \frac{a + 1}{a^2 + 1} \right) + 3 \geq 4a + 2 \left( \frac{a + 1}{a^2 + 1} \right).\]

This is equivalent to \((a - 1)^4 \geq 0\), which is clearly true.

If \(a - c = 1 - a^2c\), then this is equivalent to \((a - 1)(ac + c + 1) = 0\). Since \(a\) and \(c\) are non-negative, \(ac + c + 1\) cannot be 0, so \(a\) must equal 1 and \(c\) can be any non-negative real number. Plugging this into the inequality yields \(2c + 4 \geq 2c + 4\), which always holds true.

Since we have proved the inequality for when \(a = 0\) and \(a = b\), we are done.

**Problems with Special Conditions**

The following problems may be solved using the \(pqr\) lemma for special conditions.

**Problem 5.** Let \(a, b, c\) be non-negative real numbers such that

\[(a + b)(b + c)(c + a) = 8.\]

Prove that

\[(a + b + c)^3 + 5abc \geq 32.\]
**Problem 6.** Let $a, b, c$ be non-negative real numbers such that

$$(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = 10.$$ 

Prove that

$$\frac{9}{8} \leq \frac{a^2 + b^2 + c^2}{ab + bc + ca} \leq \frac{6}{5}. $$

**Problem 7.** Let $a, b, c$ be positive real numbers such that

$$2(a + b + c) = a^2 b + ab^2 + b^2 c + bc^2 + c^2 a + ca^2.$$ 

Prove that

$$\frac{a^2}{2 + bc} + \frac{b^2}{2 + ca} + \frac{c^2}{2 + ab} \geq 1.$$ 

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