CONTEST CORNER

SOLUIONS


CC216. Starting with a list of three numbers, the “changesum” procedure creates a new list by replacing each number by the sum of the other two. For example, from \{3, 4, 6\} “changesum” gives \{10, 9, 7\} and a new “changesum” leads to \{16, 17, 19\}. If we begin with \{20, 1, 3\}, what is the maximum difference between two numbers of the list after 2014 consecutive “changesums”?

*Originally question 23 of Irish Junior Maths Competition Final 2014.*

We received four correct solutions. We present the solution of Doddy Kastanya.

Suppose we have a list of numbers \{a, b, c\} where \(a < b < c\). The maximum difference between the largest and the smallest number is \(c - a\). The “changesum” operation on this list will create a new list \(\{b + c, a + c, a + b\}\). Since \(a < b\), we know that \(a + c < b + c\). Since \(b < c\), we also know that \(a + b < a + c\). Combining these two inequalities, we can write \(a + b < a + c < b + c\).

The maximum difference between any number is \((b + c) - (a + b)\) or \(c - a\). So, after the first “changesum” operation, the maximum difference stays the same. Since the maximum difference is independent of the order of \(\{a, b, c\}\) in the original list, the difference will always be \(c - a\).

Equipped with this knowledge, for the initial list of \{20, 1, 3\}, the maximum difference will always be 20 – 1 or 19. So, after 2014 consecutive “changesum” operations, the maximum difference between two numbers will be 19.

CC217. A right triangle \(ABC\) has its hypotenuse \(AB\) trisected at \(M\) and \(N\). If \(CM^2 + CN^2 = k \cdot AB^2\), then what is the value of \(k\)?

*Originally question 27 in the Indiana State Mathematics Contest 2009 (Comprehensive Test).*

We received eight correct submissions. We present a solution by Doddy Kastanya.

Let \(K\) and \(L\) be the two points on \(CB\) such that \(MK \perp BC\) and \(NL \perp BC\). This yields

\[
CM^2 = CK^2 + MK^2
\]

\[
CN^2 = CL^2 + NL^2.
\]

Using similar triangles \(MKB\) and \(ACB\), we have \(CK = \frac{1}{3}BC\) and \(MK = \frac{2}{3}AC\).

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Substituting this into (1), we get

\[ CM^2 = \frac{1}{9} BC^2 + \frac{4}{9} AC^2. \]

(3)

Using similar triangles \( NLB \) and \( ACB \), we have \( CL = \frac{2}{3} BC \) and \( NL = \frac{1}{3} AC \). Substituting this into (2), we get

\[ CN^2 = \frac{1}{9} AC^2 + \frac{4}{9} BC^2. \]

(4)

Adding (3) and (4) together, we get

\[ CM^2 + CN^2 = \frac{5}{9} (AB^2 + BC^2) = \frac{5}{9} AB^2. \]

So the value of \( k \) is \( \frac{5}{9} \).

Editor’s Comments. Zelator presented (and proved) the following generalization of the problem. (His solution uses the relation between a median and the sides of a triangle as well as the cosine law.)

Let \( r_1 \) and \( r_2 \) be distinct fixed positive real numbers, \( 0 < r_1 < r_2 < 1 \). Let \( ABC \) be a right triangle with the right angle at \( C \), hypotenuse \( AB \) and with lengths \( AB = c, BC = a \) and \( AB = c \). Let \( M \) and \( N \) be points on the hypotenuse \( AB \), such that \( AM = r_1c \) and \( AN = r_2c \). Finally, let \( CM = x \) and \( CN = y \).

a) Show that \( x^2 = r_1^2 a^2 + (1 - r_1)^2 b^2 \) and \( y^2 = r_2^2 a^2 + (1 - r_2)^2 b^2 \).

b) Suppose that \( r_1 + r_2 = 1 \) and \( x^2 + y^2 = kc^2 \). Show that \( k = r_1^2 + r_2^2 \).

c) Suppose that \( r_1 + r_2 \neq 1 \) and \( x^2 + y^2 = kc^2 \). Express \( k \) in terms of \( r_1, r_2 \) and the ratio \( R = \frac{b}{a} \).

CC218. Solve the following system of equations:

\[
\begin{align*}
3^\ln x &= 4^\ln y, \\
(4x)^\ln 4 &= (3y)^\ln 3.
\end{align*}
\]

Originally question 10 from the 2014 Texas A&M High School Mathematics Contest.

We received seven correct submissions. We present a solution by Šefket Šarlanagić, modified by the editor, and a generalization.

Using the fact that a logarithmic function is one-to-one and employing properties of logarithms, we get

\[
\begin{align*}
3^\ln x &= 4^\ln y, \\
(4x)^\ln 4 &= (3y)^\ln 3,
\end{align*}
\]

\( \iff \)

\[
\begin{align*}
\ln x \ln 3 &= \ln y \ln 4, \\
(ln 4 + \ln x)\ln 4 &= (\ln 3 + \ln y)\ln 3.
\end{align*}
\]

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From the first equation we see that \( \ln x = \frac{\ln y \ln 4}{\ln 3} \) and substituting into the second equation, we have

\[
\left( \ln 4 + \frac{\ln y \ln 4}{\ln 3} \right) \ln 4 = (\ln 3 + \ln y) \ln 3
\]

and so

\[
(\ln^2 4 - \ln^2 3) \left( 1 + \frac{\ln y}{\ln 3} \right) = 0.
\]

From here, since \( \ln^2 3 - \ln^2 4 \neq 0 \), we get \( \ln y = -\ln 3 \), so \( y = \frac{1}{3} \). Plugging this back into the first equation, we have

\[
\ln x = \frac{\ln y \ln 4}{\ln 3} = \frac{\ln 3 \ln 4}{\ln 3} = -\ln 4,
\]

so \( x = \frac{1}{4} \). Therefore, the solution to the system is \( (x, y) = (\frac{1}{4}, \frac{1}{3}) \).

Editor’s Comments. Sitariu and Zelator (independently) both noticed that the use of numbers 3 and 4 was arbitrary and gave the following generalized version: the system

\[
\begin{align*}
a \ln x &= b \ln y, \\
(bx)^{\ln b} &= (ay)^{\ln a},
\end{align*}
\]

has solution \( (x, y) = (\frac{1}{b}, \frac{1}{a}) \). Natural logarithms can also be replaced with logarithms in base \( c > 0 \).

**CC219.** A wooden rectangular prism has dimensions 4 by 5 by 6. This solid is painted green and then cut into 1 by 1 by 1 cubes. Find the ratio of the number of cubes with exactly two green faces to the number of cubes with exactly three green faces.

*Originally question 18 of the 2014 Texas A&M High School Mathematics Contest.*

We received three correct solutions. We present the solution of Fernando Ballesta.

There are 8 cubes with three faces coloured (which are the 8 corners), and there are

\[
4(6-2) + 4(5-2) + 4(4-2) = 16 + 12 + 8 = 36
\]

cubes with two faces coloured (which are the ones on the edges and are not corners).

So the ratio is 36 : 8 = 9 : 2, that is, for every two faces with three faces coloured there are nine with two faces coloured.

**CC220.** Two random numbers \( x \) and \( y \) are drawn independently from the closed interval \([0, 2]\). What is the probability that \( x + y > 1？\)

*Originally question 13 of the 2014 Texas A&M High School Mathematics Contest.*

We received three correct solutions. We present the solution of Doddy Kastanya.

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We can draw on the Cartesian plane the area (shaded in the figure below) represented by $x + y > 1$.

The probability of $x + y > 1$ is simply the ratio between the shaded area and the area of the overall square. The area of the non-shaded part of the square is $\frac{1}{2}$. The overall area of the square is $2 \cdot 2 = 4$. So, the area of the shaded part of the square is $4 - \frac{1}{2} = \frac{7}{2}$.

Therefore, the probability that $x + y > 1$ is $\frac{7}{2}/4$ or $\frac{7}{8}$. 

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