SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


Let $s$ be a fixed real number such that $s \geq 1$. Let $a, b, c$ and $d$ be non-negative numbers that satisfy $a + b + c + d = 4s$ and $ab + bc + cd + da + ac + bd = 6$. Express the minimum value of the product $abcd$ in terms of $s$.

The proposers provided a correct solution, given below. Two other submissions were incorrect.

We first establish that at least one of $a, b, c, d$ can vanish if and only if $4s \geq 3\sqrt{2}$.

Let $d = 0$, say. Since $(a + b + c)^2 \geq 3(ab + bc + ca)$, then $16s^2 \geq 18$. Conversely, consider the system of equations $u + 2v = 4s$, $2uv + v^2 = 6$. Eliminating $u$ yields $3v^2 - 8sv + 6 = 0$, a quadratic equation with positive real solutions iff $64s^2 \geq 72$.

With $v$ the smaller one, the quadruple $(a, b, c, d) = (u, v, v, 0)$ satisfies the conditions. Thus, the minimum value of $abcd$ is 0 when $s \geq 3\sqrt{2}/4$.

Henceforth, suppose that $1 \leq s < 3\sqrt{2}/4$. Let $abcd = p$, $abc + bcd + cda + dab = r$.

$$f(x) = \frac{1}{x}(x - a)(x - b)(x - c)(x - d) = x^3 - 4sx^2 + 6x - r + \frac{p}{x},$$

and

$$g(x) = x^2 f'(x) = 3x^4 - 8sx^3 + 6x^2 - p.$$ 

Note that $p > 0$. Since $f(x)$ has four positive roots, by Rolle’s theorem, $f'(x)$ has three positive roots (counting multiplicity). Since $g(0) < 0$, the quartic polynomial has one negative and three positive roots.

The polynomial $g'(x) = 12x(x^2 - 2sx + 1)$ has three roots, namely 0, $1/t$ and $t$, where

$$1 \leq t = s + \sqrt{s^2 - 1} < \sqrt{2}.$$ 

Since $g(x)$ has three positive roots (counting multiplicity) and $g(0) < 0$, we must have that $g(1/t) \geq 0$ and $g(t) \leq 0$. Hence

$$abcd = p \geq 3t^4 - 8st^3 + 6t^2 = 3t^4 - 4(2st - 1)t^2 + 2t^2$$

$$= 3t^4 - 4t^4 + 2t^2 = 2t^2 - t^4$$

$$= (s + \sqrt{s^2 - 1})^2[2 - (s + \sqrt{s^2 - 1})^2].$$
Let \((a, b, c, d) = (t, t, t, 2t^{-1} - t)\). Then
\[
\begin{align*}
a + b + c + d &= 2(t + t^{-1}) = 4s, \\
ab + bc + ca + db + dc &= 3t^2 + 6 - 3t^2 = 6, \text{ and} \\
abcd &= 2t^2 - t^4.
\end{align*}
\]
Therefore, when \(1 \leq s < \frac{3\sqrt{2}}{4}\), the minimum of \(abcd\) is
\[
(s + \sqrt{s^2 - 1})^2[2 - (s + \sqrt{s^2 - 1})^2],
\]
and when \(s \geq \frac{3\sqrt{2}}{4}\), the minimum of \(abcd\) is 0.

**4122. Proposed by Daniel Sitaru.**

Prove that for \(n \in \mathbb{N}\), the following holds
\[
\left(\frac{e^n - 1}{n}\right)^{2n+1} \leq \frac{(e - 1)(e^2 - 1)(e^3 - 1)\cdots(e^{2n} - 1)}{(2n)!}.
\]

We received six correct and complete solutions of which we present the one by Ángel Plaza, slightly modified by the editor.

Note that the inequality in the statement can be rewritten as
\[
\left(\frac{e^n - 1}{n}\right)^{2n+1} \leq \left(\frac{e - 1}{1}\right)\left(\frac{e^2 - 1}{2}\right)\cdots\left(\frac{e^{2n} - 1}{2n}\right).
\]

Consider the function
\[
f(x) = \ln\left(\frac{e^x - 1}{x}\right)
\]
defined for \(x > 0\) and set \(f(0) = 0\). Then \(f\) is continuous for \(x \geq 0\) and has second derivative
\[
f''(x) = \frac{(e^x - 1)^2 - x^2e^x}{x^2(e^x - 1)^2}.
\]
To show that \(f(x)\) is convex it suffices to prove that \((e^x - 1)^2 - x^2e^x > 0\). This can be reformulated to \(e^x(e^x + e^{-x} - (2 + x^2)) > 0\). But we have
\[
e^x + e^{-x} = \sum_{k=0}^{\infty} \frac{2x^{2k}}{(2k)!} > 2 + x^2.
\]
Therefore, the second derivative of \(f\) is positive and \(f(x)\) is convex for \(x > 0\). Rephrasing inequality (1) by taking logarithms we obtain
\[
(2n + 1)f(n) \leq \sum_{k=0}^{2n} f(k),
\]
which follows from Jensen’s inequality.

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4123. Proposed by Michel Bataille.

In 3-dimensional Euclidean space, a line \( \ell \) is perpendicular to the plane of the acute triangle \( A'B'C' \) at its orthocentre \( K \). Let \( A, B, C \) be the midpoints of \( B'C', C'A' \) and \( A'B' \), respectively. Show that \( BC > KA \) and if \( D \) on \( \ell \) satisfies \( KD = \sqrt{BC^2 - KA^2} \), that the tetrahedron \( ABCD \) is isosceles. (A tetrahedron is called isosceles if its opposite edges are congruent.)

We received three correct submissions and feature the solution by Leonard Giugiuc.

Without loss of generality, we choose coordinates \( A'(0, 2, 0), B'(-2u, 0, 0), \) and \( C'(2v, 0, 0) \). Because \( A' \) is acute, we can assume that \( u \) and \( v \) are positive; moreover we have \( u = \cot B' \) and \( v = \cot C' \), whence (because the triangle is acute)

\[
uv - 1 = (\cot B' + \cot C') \cot(B' + C') < 0.
\]

Consequently, \( A(v-u, 0, 0), B(v, 1, 0), C(-u, 1, 0), \) and \( K(0, 2uv, 0) \). The line \( \ell \) is therefore the set of points \( \{2, 2uv, z\} \), and

\[
BC^2 - KA^2 = (u+v)^2 - (u-v)^2 - 4u^2v^2 = 4uv(1-uv) > 0.
\]

Thus, \( BC > KA \), as claimed. We may now choose \( D = (0, 2uv, 2\sqrt{uv(1-uv)}) \).

We conclude that

\[
AD^2 = (v-u)^2 + 4u^2v^2 + 4uv(1-uv) = (u+v)^2 = BC^2;
\]

\[
BD^2 = v^2 + (2uv-1)^2 + 4uv(1-uv) = v^2 + 1 = CA^2;
\]

\[
CD^2 = u^2 + (2uv-1)^2 + 4uv(1-uv) = u^2 + 1 = AB^2.
\]

The proof is complete.

Comment (by the proposer and the third solver John Heuver). Since \( CD = AB = CA' \) and \( BD = CA = BA' \), the point \( D \) is the image of \( A' \) under a suitable rotation about axis \( BC \). Similarly, \( D \) is the image of \( B' \) and of \( C' \) under suitable rotations about axes \( CA \) and \( AB \), respectively. Thus, from an acute triangle \( A'B'C' \) and its midpoint triangle \( ABC \) drawn on cardboard, one can obtain an isosceles tetrahedron \( DABC \) by folding along \( BC, CA, AB \) till \( A', B', C' \) coincide (and naming \( D \) the point of coincidence).

4124. Proposed by George Apostolopoulos.

Let \( A_1, B_1 \) and \( C_1 \) be points on the sides \( BC, CA \) and \( AB \) of a triangle \( ABC \) such that

\[
\frac{A_1B}{A_1C} = \frac{B_1C}{B_1A} = \frac{C_1A}{C_1B} = k.
\]

Prove that

\[
\left( \frac{AA_1}{BC} \right)^2 + \left( \frac{BB_1}{CA} \right)^2 + \left( \frac{CC_1}{AB} \right)^2 \geq \left( \frac{3k}{k^2 + 1} \right)^2 \left( \frac{2r}{R} \right)^4,
\]

where \( R \) and \( r \) are the circumradius and the inradius of \( ABC \), respectively.
We received three solutions. We present the solution by Titu Zvonaru, slightly modified by the editor.

Denote by $a, b$ and $c$ the sides of the triangle. Note that $\frac{A_1B}{BC} = k$ implies that $\frac{A_1B}{k+1} = k$ and $\frac{A_1C}{BC} = 1$.

Using Stewart’s Theorem [see Editor’s Comments] for the length of a cevian, and dividing by $a^2$, we get

$$\left(\frac{AA_1}{BC}\right)^2 = \frac{k}{k+1} \cdot \frac{b^2}{a^2} + \frac{1}{k+1} \cdot \frac{c^2}{a^2} - \frac{k}{(k+1)^2}.$$

Proceed similarly to get corresponding formulae for the remaining terms on the left hand side of the desired inequality to obtain

$$\left(\frac{AA_1}{BC}\right)^2 + \left(\frac{BB_1}{AC}\right)^2 + \left(\frac{CC_1}{AB}\right)^2 = \frac{k}{k+1} \left(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\right) + \frac{1}{k+1} \left(\frac{c^2}{a^2} + \frac{a^2}{b^2} + \frac{b^2}{c^2}\right) - \frac{3k}{(k+1)^2}. \quad (1)$$

By the AM-GM inequality, $\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} \geq 3$ and $\frac{c^2}{a^2} + \frac{a^2}{b^2} + \frac{b^2}{c^2} \geq 3$ (in both cases, with equality when $a = b = c$), so from (1) we get

$$\left(\frac{AA_1}{BC}\right)^2 + \left(\frac{BB_1}{AC}\right)^2 + \left(\frac{CC_1}{AB}\right)^2 \geq \frac{3k}{k+1} + \frac{3}{k+1} - \frac{3k}{(k+1)^2} = 3 - \frac{3k}{(k+1)^2}.$$

We now prove that $3 - \frac{3k}{(k+1)^2} \geq \left(\frac{3k}{k^2+1}\right)^2$, or equivalently $\frac{3k^2}{(k^2+1)^2} + \frac{k}{(k+1)^2} \leq 1$.

Using AM-GM, we have

$$k^2 + 1 \geq 2k, \quad \text{which we can rearrange to} \quad \frac{3k^2}{(k^2+1)^2} \leq \frac{3}{4}; \quad \text{and} \quad k + 1 \geq 2\sqrt{k}, \quad \text{which we can rearrange to} \quad \frac{k}{(k+1)^2} \leq \frac{1}{4}.$$ 

It follows that $\frac{3k^2}{(k^2+1)^2} + \frac{k}{(k+1)^2} \leq 1$ (equality holds when $k = 1$). So far we have

$$\left(\frac{AA_1}{BC}\right)^2 + \left(\frac{BB_1}{AC}\right)^2 + \left(\frac{CC_1}{AB}\right)^2 \geq \left(\frac{3k}{k^2+1}\right)^2.$$

Note that Euler’s inequality, $R \geq 2r$, implies $1 \geq \left(\frac{2r}{R}\right)^4$, which concludes the proof of the desired inequality. Equality holds when $a = b = c$ and $k = 1$.

Editor’s Comments: Stewart’s Theorem gives us a formula for calculating the length of a cevian in a triangle: given a cevian $AA_1$ with $\frac{AA_1}{BC} = m$ and $\frac{A_1C}{BC} = n$, we have

$$(AA_1)^2 = mb^2 + nc^2 - mna^2.$$

The formula can also be derived easily from the cosine law.

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Start with a triangle $A_1A_2A_3$ in the Euclidean plane and three nonzero real numbers $\ell_1, \ell_2, \ell_3$. Define $M_k$ and $C_k$ to be points on the line $A_{k+1}A_{k+2}$ such that

$$\frac{A_{k+1}M_k}{M_kA_{k+2}} = \ell_k$$

and

$$C_kM_{k+1}A_{k+1}, \quad k = 1, 2, 3$$

(with subscripts reduced modulo 3 and distances taken to be signed, so that $M_k$ is between $A_{k+1}$ and $A_{k+2}$ precisely when $\ell_k$ is positive).

Denote by $R_k$ the point where $C_kM_{k+1}$ intersects $C_{k+1}M_k$, $k = 1, 2, 3$. Show that

$$\begin{vmatrix} R_1 & R_2 & R_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \ell_2 \frac{A_2}{A_1} + \ell_3 \frac{A_3}{A_1} - \ell_1 \frac{A_3}{A_1} \ell_2 \frac{A_2}{A_1}$$

where square brackets denote area.

We received three correct submissions and feature the solution by AN-anduud Problem Solving Group.

From $\triangle R_3R_1R_2 \sim \triangle A_1A_2A_3$ we get

$$\frac{|R_1R_2R_3|}{|A_1A_2A_3|} = \left( \frac{R_3R_1}{A_1A_2} \right)^2.$$  \hfill (1)

Of course,

$$\frac{R_3R_1}{A_1A_2} = \frac{M_2C_1 - M_2R_3 - R_1C_1}{A_1A_2} = \frac{M_2C_1}{A_1A_2} - \frac{M_2R_3}{A_1A_2} - \frac{R_1C_1}{A_1A_2}.$$  \hfill (2)

Because $\triangle A_1A_2A_3 \sim \triangle M_2C_1A_3$ we get

$$\frac{M_2C_1}{A_1A_2} = \frac{M_2A_3}{A_1A_3} = \frac{M_2A_3}{A_1M_2 + M_2A_3} = \frac{M_2A_3}{A_1A_2} \frac{1}{1 + \frac{M_2A_3}{A_1A_2}} = \frac{\ell_2}{1 + \ell_2}.$$  \hfill (3)
From $M_2R_3 = A_1C_3$ and $A_1A_3 \parallel C_3M_1$ we have

$$\frac{M_2R_3}{A_1A_2} = \frac{A_1C_3}{A_1A_2} = \frac{A_1C_3}{A_1C_3 + C_3A_2} = \frac{1}{1 + \frac{C_3A_2}{A_1C_3}} = \frac{1}{1 + \ell_1}. \quad (4)$$

Finally, $R_1C_1 = M_3A_2$, hence we get

$$\frac{R_1C_1}{A_1A_2} = \frac{M_3A_2}{A_1M_3 + M_3A_2} = \frac{1}{1 + \frac{M_3M_1}{M_3A_2}} = \frac{1}{1 + \ell_3}. \quad (5)$$

The desired result follows immediately from equations (1)–(5).

4126. Proposed by Mihaela Berindeanu.

Let $ABC$ be an acute-angled triangle. Prove that

$$\sum_{\text{cyc}} \tan \frac{A}{2} \tan \frac{B}{2} \geq \sqrt{3}.$$  

We received 22 solutions, all correct. Titu Zvonaru contributed five of them, and Arkady Alt supplied another two. Of the many different approaches, the solutions that exploited Jensen’s inequality were the shortest and most popular. We feature the solution from Prithwijit De, which is typical of that approach.

Our argument is valid for all proper triangles, so we drop the restriction to acute angles.

We know that for all triangles $ABC$,

$$\tan(A/2) \tan(B/2) + \tan(B/2) \tan(C/2) + \tan(C/2) \tan(A/2) = 1.$$ 

Let

$$x = \tan(A/2) \tan(B/2),$$

$$y = \tan(B/2) \tan(C/2),$$

$$z = \tan(C/2) \tan(A/2).$$

Then we have to show that

$$\sum_{\text{cyc}} \frac{x}{\sqrt{1-x}} \geq \sqrt{3},$$

subject to the conditions $x + y + z = 1$ and $0 < x, y, z < 1$. Now

$$\frac{x}{\sqrt{1-x}} = \frac{1}{\sqrt{1-x}} - \sqrt{1-x},$$

and the function $f(x) = \sqrt{1-x}$ is continuous and concave on $(0, 1)$. Therefore the function

$$h(x) = \frac{1}{\sqrt{1-x}} - \sqrt{1-x}$$

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is continuous and convex on $(0, 1)$. Thus by Jensen’s inequality we obtain
\[
\sum_{\text{cyc}} \frac{x}{\sqrt{1-x}} \geq 3 \left( \frac{x+y+z}{3} \right) = \sqrt{\frac{3}{2}}.
\]
as desired.

Equality holds if and only if $x = y = z$, which is equivalent to $\Delta ABC$ being equilateral.

4127. Proposed by D. M. Bătinețu-Giurgiu and Neculai Stanciu.

Calculate
\[
\lim_{n \to \infty} \int_{\sqrt{n^n}}^{\sqrt{n+1)!} f \left( \frac{x}{n} \right) dx,
\]
where $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a continuous function.

We received 8 solutions. We present the solution by Leonard Giugiuc, slightly modified by the editor.

Perform the substitution $t = \frac{x}{n}$ to get
\[
\int_{\sqrt{n^n}}^{\sqrt{n+1)!}} f \left( \frac{x}{n} \right) dx = n \int_{\sqrt{n^n}}^{\sqrt{n+1)!}} f(t) dt.
\]

By the mean value theorem, there exists a $c_n$ with
\[
\frac{\sqrt{n!}}{n} < c_n < \frac{n+1)!}{n},
\]
such that
\[
\int_{\sqrt{n^n}}^{\sqrt{n+1)!}} f(t) dt = \left( \frac{n+1)!}{n} \sqrt{n^n} - \frac{\sqrt{n!}}{n} \right) f(c_n).
\]

It follows that
\[
\lim_{n \to \infty} \int_{\sqrt{n^n}}^{\sqrt{n+1)!}} f \left( \frac{x}{n} \right) dx = \lim_{n \to \infty} n \left( \frac{n+1)!}{n} \sqrt{n^n} - \frac{\sqrt{n!}}{n} \right) f(c_n)
\]
\[
= \lim_{n \to \infty} \left( \frac{n+1)!}{n} - \sqrt{n!} \right) \cdot \lim_{n \to \infty} f(c_n),
\]
assuming we can show that both limits exist, which we now proceed to do.

From Stirling’s approximation for factorials we know that for all positive integers $n$
\[
\sqrt{2 \pi n} \cdot n^n \cdot e^{-n} \leq n! \leq \sqrt{2 \pi n} \cdot n^n \cdot e^{-n} \cdot \frac{1}{\sqrt{2\pi n}}.
\]

This enables us to show (using the squeeze theorem and standard limit techniques) that
\[
\lim_{n \to \infty} \frac{\sqrt{n!}}{n} = \frac{1}{e}.
\]
It follows that
\[
\lim_{n \to \infty} \frac{\sqrt[n+1]{(n+1)!}}{n} = \lim_{n \to \infty} \frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \frac{n+1}{n} = \frac{1}{e}
\]
as well, whence by the squeeze theorem applied to the interval of \(c_n\) we conclude that \(\lim_{n \to \infty} c_n = \frac{1}{e}\). Hence, since \(f\) is continuous, \(\lim_{n \to \infty} f(c_n) = f\left(\frac{1}{e}\right)\).

Using Stirling’s approximation again, one can also show that
\[
\lim_{n \to \infty} \left( \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) = \frac{1}{e}.
\]

Therefore, from (1), it follows that
\[
\lim_{n \to \infty} \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f \left( \frac{x}{n} \right) \, dx = \frac{1}{e} \cdot f\left(\frac{1}{e}\right).
\]

**4128. Proposed by Valcho Milchev and Tsvetelina Karamfilova.**

Let \(A_n\) be the number of domino tilings of a rectangular \(3 \times 2n\) grid. Let \(L(2n, 2k)\) be the number of domino tilings of the grid composed of two rectangular grids of dimensions \(3 \times 2n\) and \(3 \times 2k\) with \(n \geq 2\) and \(k \geq 1\) (depicted below):

Prove that \(L(2n, 2n) = A_{2n}\).

We received three correct solutions. We present the solution by Oliver Geupel.

Let \(G\) be a grid composed of a \(3 \times 3\) grid \(B\), a \(3 \times (2n - 3)\) grid \(C\) above \(B\) and a \(2n \times 3\) grid \(D\) to the right of \(B\). Also, let \(H\) be a grid composed of grids \(B\) and \(C\) and a \(3 \times 2n\) grid \(E\) below \(B\). We have to show that \(G\) and \(H\) admit the same number of domino tilings.

Suppose we have a domino tiling of \(G\). Consider the dominoes that cover a cell in both \(B\) and \(C\). As the number of cells in \(C\) is odd, there have to be one or three such dominoes. Furthermore, if there is only one such domino then it cannot be in the middle column, as can be seen by a chessboard colouring argument (colour the cells of \(C\) minus the bottom cell of the middle column alternatingly white and black; the number of white cells differs from the number of black cells by two, but...
any domino in a tiling covers one white and one black cell). Similarly we consider dominoes of the tiling that cover a cell in both $B$ and $D$. As the number of cells in $D$ is even, there must be zero or two such dominoes and, again by a chessboard colouring argument, if there are two such dominoes then one must be in the middle column.

We obtain the following cases for tilings of $G$, where $B'$, $C'$, and $D'$ are the subgrids of $G$ formed by the dominoes that only cover cells of $B$, $C$, or $D$, respectively.

![Diagram](image1)

For domino tilings of $H$, we can make the same argument as above to obtain the following cases in terms of $B'$, $C'$, and $E'$, the subgrids of $H$ formed by the dominoes that only cover cells of $B$, $C$, and $E$, respectively.

![Diagram](image2)

Comparing shapes of $B'$, $C'$, and $D'$ for $G$ with the shapes of $B'$, $C'$, and $E'$ for $H$ we see that cases $G1$ and $H1$ admit the same number of tilings, as do cases $G2$-$G5$ and $H2$-$H5$.

In the case $G6$ the grid $B'$ admits three tilings, which is the same as the number of tilings of $B'$ in cases $H6$ and $H7$ combined. Hence the number of tilings in case $G6$ equals those of cases $H6$ and $H7$ combined.

Similarly, case $G7$ admits the same number of tilings as cases $H8$ and $H9$ combined.

In conclusion, we obtain that the number of tilings of $G$ is the same as the number of tilings of $H$.

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4129. Proposed by Lorean Saceanu.

Let $\triangle ABC$ be an acute-angle triangle and let $\gamma = 3(2 - \sqrt{3})$. Prove that

$$\sec A + \sec B + \sec C \geq \gamma + \tan A + \tan B + \tan C.$$ 

We received 12 submissions all of which were correct. Except for one, all of them are very similar to one another so we will present a composite of these solutions.

Let $f(x) = \sec x - \tan x$, $x \in (0, \frac{\pi}{2})$. Then we have $f'(x) = \sec x \tan x - \sec^2 x$ and $f''(x) = \sec^3 x + \sec x \tan^2 x - 2 \sec^2 x \tan x = (\sec x)(\sec x - \tan x)^2 > 0$,

so $f(x)$ is strictly convex on $(0, \frac{\pi}{2})$.

Since $A, B, C \in (0, \frac{\pi}{2})$ such that $A + B + C = \pi$, we have, by Jensen’s Inequality that

$$\sec A + \sec B + \sec C - \tan A - \tan B - \tan C = f(A) + f(B) + f(C) \geq 3f\left(\frac{A+B+C}{3}\right) = 3f\left(\frac{\pi}{3}\right) = 3(\sec \frac{\pi}{3} - \tan \frac{\pi}{3}) = 3(2 - \sqrt{3}) = \gamma.$$ 

This completes the proof. It is easy to see that equality holds if and only if $A = B = C = \frac{\pi}{3}$; i.e. if and only if $\triangle ABC$ is equilateral.


Let $a, b$ and $c$ be nonnegative real numbers such that $a + b + c = ab + bc + ca > 0$. Prove that

$$\sqrt[n]{a} + \sqrt[n]{b} + \sqrt[n]{c} \geq 2 \sqrt[n]{2}$$

for any integer $n \geq 3$ and determine the case for equality to hold.

Correct solutions were received from the proposer and Arđak Mirzakhmedov. There was one incorrect submission. We present the solution of Mirzakhmedov.

Solution. The condition implies that at most one of $a, b, c$ can vanish. Since

$$2 = \frac{2(ab + bc + ca)}{a + b + c} \leq \frac{2ab}{a + b} + \frac{2bc}{b + c} + \frac{2ca}{c + a} \leq \sqrt{ab} + \sqrt{bc} + \sqrt{ca},$$

with equality iff $(a, b, c) = (0, 2, 2), (2, 0, 2), (2, 2, 0)$, it is enough to prove that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq 2 \cdot \sqrt[3]{\sqrt{ab} + \sqrt{bc} + \sqrt{ca}}.$$ 

Let $(a, b, c) = (u^{2n}, v^{2n}, w^{2n})$. The desired inequality becomes

$$(u^2 + v^2 + w^2)^n \geq 2^n(u^n v^n + v^n w^n + w^n u^n).$$

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We establish this by induction. When \( n = 3 \), the inequality follows from
\[
(u^2 + v^2 + w^2)^3 - 8(u^3v^3 + v^3w^3 + w^3u^3)
= [u^2(u^2 - v^2)(u^2 - w^2) + v^2(v^2 - w^2)(v^2 - u^2) + w^2(w^2 - u^2)(w^2 - v^2)]
+ 4[u^2v^2(u - v)^2 + v^2w^2(v - w)^2 + w^2u^2(w - u)^2]
+ 3u^2v^2w^2 \geq 0,
\]
the first term in square brackets being nonnegative by Schur’s inequality.

Suppose that the inequality holds for \( n \geq 3 \). Then
\[
(u^2 + v^2 + w^2)^{n+1} \geq 2^n(u^2 + v^2 + w^2)(u^n v^n + v^n w^n + w^n u^n)
\geq 2^n[(u^{n+2}v^n + u^n v^{n+2}) + (v^{n+2}w^n + v^n w^{n+2})
+ (w^{n+2}u^n + w^n u^{n+2})]
\geq 2^{n+1}(u^{n+1}v^{n+1} + v^{n+1}w^{n+1} + w^{n+1}u^{n+1}),
\]
by the arithmetic-geometric means inequality.

Editor’s Comment. The proposer has pointed out that this technique can be used for other problems, such as in proofs of the following inequalities for \( n \geq 3 \) and \( m > 0 \):
\[
\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \geq 2 \cdot \sqrt[3]{2};
\sqrt[4]{ab} + \sqrt[4]{bc} + \sqrt[4]{ca} \geq \sqrt[4]{4};
\sqrt[2]{ab} + \sqrt[2]{bc} + \sqrt[2]{ca} \geq \sqrt[2]{2};
(ab)^m + (bc)^m + (ca)^m \geq \min\{4^m, 3\}.
\]
In each case we begin in the same way. Wolog, let \( a \leq b \leq c \). Since we have that
\((a + b + c)^2 \geq 3(ab + bc + ca)\), then \( a + b + c = ab + bc + ca \geq 3 \) and \( bc \geq 1 \). Let
\( b + c = 2s \) and \( bc = p^2 \), so that \( s \geq p \geq 1 \) and \( a = (2s - p^2)(2s - 1)^{-1} \). From
the arithmetic-geometric means inequality, we have that \( b^m + c^m \geq 2p^m \), in particular for
\( m = 1/n \).

If \( p \geq 2 \), then all four inequalities hold easily, with equality when \( (a, b, c) = (0, 2, 2) \).
The hard part is the case \( 1 \leq p < 2 \). Since
\[
(2s - p^2)(2s - 1)^{-1} \geq (2p - p^2)(2p - 1)^{-1},
\]
it is enough to establish the inequalities with \( a \) replaced by \( (2p - p^2)(2p - 1)^{-1} \) and
\( b^m + c^m \) replaced by \( 2p^m \). At this point, the computations become rather complicated. We feel that there ought to be a more elegant way to end these proofs! Can any reader supply one?