THE OLYMPIAD CORNER

No. 351
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The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by November 1, 2017.

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OC321. Solve in positive integers

\[ x^y y^z = (x + y)^z. \]

OC322. Let \( a, b, c \in \mathbb{R}^+ \) such that \( abc = 1 \). Prove that

\[ a^2 b + b^2 c + c^2 a \geq \sqrt{(a + b + c)(ab + bc + ca)}. \]

OC323. Let \( ABC \) be a triangle. \( M \), and \( N \) points on \( BC \), such that \( BM = CN \), with \( M \) in the interior of \( BN \). Let \( P \) and \( Q \) be points in \( AN \) and \( AM \) respectively such that \( \angle PMC = \angle MAB \), and \( \angle QNB = \angle NAC \). Prove that \( \angle QBC = \angle PCB \).

OC324. Given an integer \( n > 1 \) and its prime factorization \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \), define a function

\[ f(n) = \alpha_1 p_1^{\alpha_1 - 1} \alpha_2 p_2^{\alpha_2 - 1} \cdots \alpha_k p_k^{\alpha_k - 1}. \]

Prove that there exist infinitely many integers \( n \) such that \( f(n) = f(n - 1) + 1 \).

OC325. Let \( S = \{1, 2, \ldots, n\} \), where \( n \geq 1 \). Each of the \( 2^n \) subsets of \( S \) is to be coloured red or blue. (The subset itself is assigned a colour and not its individual elements.) For any set \( T \subseteq S \), we then write \( f(T) \) for the number of subsets of \( T \) that are blue.

Determine the number of colourings that satisfy the following condition: for any subsets \( T_1 \) and \( T_2 \) of \( S \),

\[ f(T_1) f(T_2) = f(T_1 \cup T_2) f(T_1 \cap T_2). \]
OC321. Déterminer les solutions entières strictement positives de l’équation\[
x^y y^z = (x + y)^z.\]

OC322. Soit \( a, b \) et \( c \) des réels strictement positifs tels que \( abc = 1 \). Démontrer que\[
a^2 b + b^2 c + c^2 a \geq \sqrt{(a + b + c)(ab + bc + ca)}.\]

OC323. Soit un triangle \( ABC \). Soit \( M \) et \( N \) des points sur \( BC \) tels que \( BM = CN \) et que \( M \) soit sur le segment \( BN \). Soit \( P \) et \( Q \) des points sur les segments respectifs \( AN \) et \( AM \) tels que \( \angle PMC = \angle MAB \) et \( \angle QNB = \angle NAC \). Démontrer que \( \angle QBC = \angle PCB \).

OC324. Soit un entier \( n, n > 1 \), et sa factorisation première \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \). On définit une fonction \( f \) comme suit:\[
f(n) = \alpha_1 p_1^{\alpha_1-1} \alpha_2 p_2^{\alpha_2-1} \cdots \alpha_k p_k^{\alpha_k-1}.
\]
Démontrer qu’il existe une infinité de valeurs de \( n \) pour lesquelles\[
f(n) = f(n - 1) + 1.
\]

OC325. Soit \( S = \{1, 2, \ldots, n\} \) \((n \geq 1)\). On veut colorer chacun des \( 2^n \) sous-ensembles de \( S \) en rouge ou en bleu. (Chaque sous-ensemble reçoit une couleur et non pas ses éléments.) Étant donné un sous-ensemble \( T \) de \( S \), \( f(T) \) représente le nombre de sous-ensembles de \( T \) qui sont bleus.
Démontrer le nombre de coloriages qui satisfont à la condition suivante:
\[
f(T_1) f(T_2) = f(T_1 \cup T_2) f(T_1 \cap T_2),
\]
pour tous sous-ensembles \( T_1 \) et \( T_2 \) de \( S \).


**OLYMPIAD SOLUTIONS**


**OC261.** Show that there are no 2-tuples \((x, y)\) of positive integers satisfying the equation \((x + 1)(x + 2)\cdots(x + 2014) = (y + 1)(y + 2)\cdots(y + 4028)\).

*Originally problem 3 from day 1 of the 2014 China Team Selection Test.*

No submitted solutions.

**OC262.** In obtuse triangle \(ABC\), with the obtuse angle at \(A\), let \(D, E, F\) be the feet of the altitudes through \(A, B, C\) respectively. \(DE\) is parallel to \(CF\), and \(DF\) is parallel to the angle bisector of \(\angle BAC\). Find the angles of the triangle.

*Originally problem 3 of the 2014 South Africa National Olympiad.*

We received 5 correct submissions and 1 incorrect submission. We present the solution by Michel Bataille.

We show that \(\angle A = \frac{3\pi}{5}, \angle B = \frac{\pi}{10}, \angle C = \frac{3\pi}{10}\).

Let \(BC = a, CA = b, AB = c\), as usual. Since \(DE \parallel CF\), the line \(DE\) intersects \(AB\) orthogonally, say at \(K\). Let \(U\) be the foot of the internal bisector of \(\angle BAC\).

We know that \(\frac{BU}{c} = \frac{UC}{b} = \frac{a}{b + c}\); in addition, since \(AU \parallel DF\), we have \(\frac{BD}{DF} = \frac{BD}{BF}\), hence \(BD = \frac{a}{b + c} BF\).

Also \(\frac{BD}{c} = \frac{BF}{a} (= \cos B)\) so that \(BD = \frac{c}{a} BF\). This yields \(\frac{a}{b + c} = \frac{c}{a}\), that is, \(bc = a^2 - c^2\). It follows that \(\sin B \sin C = \sin^2 A - \sin^2 C\), which successively
rewrites as
\[ \sin B \sin C = (\sin A - \sin C)(\sin A + \sin C), \]
\[ \sin B \sin C = 2 \sin \frac{A - C}{2} \cos \frac{A + C}{2} \cdot 2 \sin \frac{A + C}{2} \cos \frac{A - C}{2}, \]
\[ \sin B \sin C = \sin(A - C) \sin(A + C). \]

Since \( \sin(A + C) = \sin B \) and \( A - C \) and \( C \) are acute, we obtain \( A - C = C \). Thus, \( A = 2C \) and \( B = \pi - 3C \).

Now, let \( H \) be the orthocentre of \( \Delta ABC \). Clearly, \( D \) and \( E \) are on the circle with diameter \( HC \), hence the trapezoid \( CHED \) is isosceles and \( HE = DC \).

Then the right-angled triangles \( AEH \) and \( ADC \), which obviously are similar, are congruent and so \( AE = AD \). It follows that \( BA \) is the angle bisector of \( \angle CBH \) and \( \angle HBF = B = \pi - 3C \). Since we also have \( \angle BAE = \pi - A = \pi - 2C \), we obtain that
\[ \angle HBF = \angle EBA = \frac{\pi}{2} - (\pi - 2C) = 2C - \frac{\pi}{2}. \]

We conclude that \( \pi - 3C = 2C - \frac{\pi}{2} \), hence
\[ C = \frac{3\pi}{10}, \quad A = 2C = \frac{3\pi}{5}, \quad B = \pi - 3C = \frac{\pi}{10}. \]

**OC263.** An integer \( n \geq 3 \) is called *special* if it does not divide
\[ (n - 1)! \left(1 + \frac{1}{2} + \ldots + \frac{1}{n-1}\right). \]

Find all special numbers \( n \) such that \( 10 \leq n \leq 100 \).

*Originally problem 5 from day 2 of the 2014 Argentine National Olympiad.*

*We received 2 correct submissions. We present the solution by Oliver Geupel.*

We prove the following claims:

1. Every odd number is not special.
2. For every prime number \( p \), the number \( n = 2p \) is special.
3. For every prime number \( p \geq 3 \), the number \( n = 2p^2 \) is not special.
4. For every prime \( p \) and every integer \( s \geq 3 \), \( n = 2p^s \) is not special.
5. For any coprime integers \( q \geq 2 \) and \( r \geq 2 \), \( n = 2qr \) is not special.

As a consequence, the desired numbers are
\[ 10, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, \text{ and } 94. \]
Proofs. If \( n \) is odd, then

\[
(n-1)! \sum_{k=1}^{n-1} \frac{1}{k} = \sum_{k=1}^{(n-1)/2} \frac{(n-1)!}{k(n-k)} ((n-k) + k) = n \sum_{k=1}^{(n-1)/2} \frac{(n-1)!}{k(n-k)}
\]

is divisible by \( n \), which proves (1).

Next, let \( n \) be even, say, \( n = 2m \). Then,

\[
(n-1)! \sum_{k=1}^{n-1} \frac{1}{k} = n \sum_{k=1}^{(n-1)/2} \frac{(n-1)!}{k(n-k)} + \frac{(n-1)!}{n/2};
\]

whence \( n \) is special if and only if \( 2m^2 \nmid (2m-1)! \). Since for every prime \( p \) it holds \( p^2 \nmid (2p-1)! \), the claim (2) follows.

For every prime \( p \geq 3 \), we have \( p < 2p < p^2 < 2p^2 - 1 \); hence \( 2p^4 \mid (2p^2 - 1)! \), which proves (3).

Let \( p \geq 3 \) be a prime number and \( s \geq 3 \) be a natural number. Then

\[
2 < p < p^2 < \cdots < p^s \leq 2p^s - 1.
\]

Thus, \( 2p^{s(s+1)/2} = 2p^{1+2+\cdots+s} \) divides \( (2p^s - 1)! \), so that

\[
2p^{2s} \mid 2p^{s(s+1)/2} \mid (2p^s - 1)!.
\]

This proves (4) for \( p \geq 3 \).

For \( s \geq 4 \), we have \( 2 < 2^2 < \cdots < 2^s < 2^{s+1} - 1 \), which implies

\[
2^{2s+1} \mid 2^{s(s+1)/2} \mid (2^{s+1} - 1)!.
\]

Moreover, 16 is not special by inspection. We have proven (4) for \( p = 2 \).

Let \( q \geq 2 \) and \( r \geq 2 \) be coprime numbers. Then, \( q, 2q, r, \) and \( 2r \) are distinct numbers which are less than \( 2qr - 1 \). Consequently, \( 2q^2r^2 \mid (2qr - 1)! \). This proves (5) and completes the solution.

**OC264.** A positive integer is called beautiful if it can be represented in the form \( x^2 + y^2 \) for two distinct positive integers \( x, y \). A positive integer that is not beautiful is ugly.

1. Prove that 2014 is a product of a beautiful number and an ugly number.

2. Prove that the product of two ugly numbers is also ugly.

*Originally problem 4 from day 2 of the 2014 Indonesia Mathematical Olympiad.*

We received 2 correct submissions and 1 incorrect submission. We present the solution by Steven Chow.
From the Lemmas below, it follows that 2 × 19 is ugly and 53 is beautiful, so
2014 = (2)(19)(53) is the product of a beautiful number and an ugly number.
Alternatively, note that
\[ 2014 = \frac{1330^2 + 2394^2}{1330 + 2394} \]
and thus 2014 is beautiful. Part 1 is then proven since 1 is ugly and 2014 = 1·2014.

**Lemma 1** For all integers \( n \geq 1 \), \( n \) is beautiful if and only if \( 2n \) is beautiful.

**Proof.** If \( n = \frac{x^2+y^2}{x+y} \) for some distinct integers \( x, y \geq 1 \), then \( 2n = \frac{(2x)^2+(2y)^2}{2(x+y)} \).

If \( 2n = \frac{x^2+y^2}{x+y} \) for some distinct integers \( x, y \geq 2 \), then \( x^2 + y^2 \equiv 0 \pmod{2} \), so \( (x, y) \not\in \{(0,0), (1,1)\} \pmod{2} \).

If \( (x, y) \equiv (1, 1) \pmod{2} \), then \( x^2 + y^2 \equiv 1 \pmod{4} \) and \( \frac{x^2+y^2}{x+y} \not\equiv 0 \pmod{2} \), so \( x = y \equiv 0 \pmod{2} \).

Therefore, for some distinct integers \( x_1, y_1 \geq 1 \), \( 2n = \frac{x_1^2+y_1^2}{x_1+y_1} \). Thus \( n = \frac{x_1^2+y_1^2}{x_1+y_1} \).

**Lemma 2** For all integers \( n \geq 1 \) such that \( n \equiv 1 \pmod{2} \), \( n \) is beautiful if and only if \( 2n^2 \) is a sum of 2 distinct square numbers.

**Proof.** If \( n = \frac{x^2+y^2}{x+y} \) for some distinct integers \( x, y \geq 1 \), then \( (2x-n)^2+(2y-n)^2 = 2n^2 \).

If \( 2n^2 = a^2+b^2 \) for some positive integers \( a \neq b \), then \( a^2+b^2 = 2n^2 \equiv 2 \pmod{4} \). Thus, \( a \equiv b \equiv 1 \pmod{2} \), so \( \frac{a+n}{2} \) and \( \frac{b+n}{2} \) are distinct positive numbers and
\[
    n = \left( \frac{a+n}{2} \right)^2 + \left( \frac{b+n}{2} \right)^2 = \frac{a+b+n}{2}.
\]

**Lemma 3** Let \( k \geq 1 \) be any integer. Let \( p_j \equiv 3 \pmod{4} \) be any prime, for all integers \( 1 \leq j \leq k \). Then \( 2(\prod_{i=1}^{k} p_j)^2 \) is not the sum of 2 distinct square numbers.

**Proof.** Assume for the sake of contradiction that there exist positive integers \( a \neq b \) such that \( a^2 + b^2 = 2(\prod_{i=1}^{k} p_j)^2 \).

If there does not exist an integer \( 1 \leq m \leq k \) such that \( p_m \mid a \) and \( p_m \mid b \), then \( p_m \mid a \) or \( p_m \mid b \) for all \( 1 \leq m \leq k \). Hence, \( a = b = \prod_{i=1}^{k} p_j \) which is a contradiction.

Thus, there exists an integer \( 1 \leq m \leq k \) such that \( p_m \mid a \) and \( p_m \mid b \). This implies that there exists an integer \( 1 \leq u \leq p_m - 1 \) such that \( au \equiv b \pmod{p_m} \), so
\[
0 \equiv a^2 + b^2 \equiv a^2(1 + u^2) \pmod{p_m} \implies u^2 \equiv -1 \pmod{p_m} \implies \left( \frac{u}{p_m} \right) = 1.
\]
But \( p_m \equiv 3 \pmod{4} \), so this is a contradiction.

**Lemma 4** Let \( c \geq 1 \) be any integer such that \( c \equiv 1 \pmod{2} \). Let \( p \equiv 1 \pmod{4} \) be any prime. Then \( 2(cp)^2 \) is the sum of two distinct square numbers.
Proof. It is well known that $p$ is the sum of two square numbers. Let $a > b \geq 1$ be the integers such that $a^2 + b^2 = p$. Since $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2) = 2p$, we have

$$2p^2 = p(2p) = (a^2 + b^2)((a + b)^2 + (a - b)^2)$$
$$= (a(a + b) + b(a - b))^2 + (a(a - b) - b(a + b))^2$$
$$= (a^2 + 2ab - b^2)^2 + (a^2 - 2ab - b^2)^2.$$

Thus $2(cp)^2 = ((a^2 + 2ab - b^2)c)^2 + ((a^2 - 2ab - b^2)c)^2$. $\square$

The number 1 is an ugly number. From Lemmas 1–4, for all integers $n \geq 2$, $n$ is beautiful if and only if $n$ is divisible by a prime that is congruent to 1 (mod 4). Therefore, the product of two ugly numbers is also ugly.

**OC265.** Five airway companies operate in a country consisting of 36 cities. Between any pair of cities exactly one company operates two way flights. If some air company operates between cities $A, B$ and $B, C$ we say that the triple $A, B, C$ is properly-connected. Determine the largest possible value of $k$ such that no matter how these flights are arranged there are at least $k$ properly-connected triples.

*Originally problem 6 from day 2 of the 2014 Turkey Mathematical Olympiad.*

*No solutions were submitted.*