THE CONTEST CORNER

No. 53

John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by November 1, 2017.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC261. Peter is walking through a train tunnel when he hears a train approaching. He knows that on this section of track trains travel at 60 mph. The tunnel has equally spaced marker posts, with post 0 at one end and post 12 at the other end. Peter is by post 7 when he hears the train. He quickly works out that whether he runs to the nearer end or the further end of the tunnel as fast as he can (at constant speed) he will just exit the tunnel before the train reaches him. How fast can Peter run?

CC262. In the diagram, the square has two of its vertices on the circle of radius 1 unit and the other two vertices lie on a tangent to the circle. Find the area of the square.

CC263. An old fashioned tram starts from the station with a certain number of men and women on board. At the first stop, a third of the women get out and their places are taken by men. At the next stop, a third of the men get out and their places are taken by women. There are now two more women than men and as many men as there originally were women. How many men and women were there on board at the start?

CC264. Kirsty runs three times as fast as she walks. When going to school one day she walks for twice the time she runs and the journey takes 21 minutes. The next day she follows the same route but runs for twice the time she walks. How long does she take to get to school?
CC265. In the diagram, $ST$ is parallel to $QR$, $UT$ is parallel to $SR$, $PU = 4$ and $US = 6$. Find the length of $SQ$.

CC261. Pierre marche dans un tunnel de chemin de fer lorsqu’il entend venir un train. Il sait que sur cette section du chemin de fer, le train se déplace à une vitesse de 60 km/h. Des bornes numériques équidistantes sont placées dans le tunnel, la borne 0 étant placée à l’entrée et la borne 12 étant à l’autre extrémité. Pierre est vis-à-vis la borne 7 lorsqu’il entend le train. Il calcule rapidement que s’il court à toute vitesse (constante) vers l’une ou l’autre extrémité du tunnel, il réussira à sortir du tunnel juste avant l’arrivée du train. À quelle vitesse Pierre peut-il courir?

CC262. Dans la figure suivante, deux sommets du carré sont situés sur le cercle de rayon 1 et ses deux autres sommets sont situés sur une tangente au cercle. Déterminer l’aire du carré.

CC263. Un tramway quitte une station avec à bord un nombre quelconque d’hommes et de femmes. Au premier arrêt, un tiers des femmes descendent et leurs places sont prises par des hommes qui arrivent. Au deuxième arrêt, un tiers des hommes descendent et leurs places sont prises par des femmes qui arrivent. Il y a maintenant deux femmes de plus que d’hommes à bord. De plus, le nombre d’hommes est le même que le nombre initial de femmes à bord. Combien y avait-il d’hommes et de femmes à bord au départ?

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CC264. Kim court trois fois plus vite qu’elle ne marche. Un jour, en se rendant à l’école, elle met deux fois plus de temps à marcher qu’à courir et complète le trajet en 21 minutes. Le lendemain, elle prend le même chemin, mais elle met deux fois plus de temps à courir qu’à marcher. Combien de temps met-elle pour se rendre à l’école?


![Diagram](image-url)
CONTEST CORNER
SOLUTIONS


CC211. A rectangular sheet of paper whose dimensions are 12 × 18 is folded along a diagonal, which creates the M-shaped region drawn at the right. Find the area of the shaded region.

![Diagram of a rectangular sheet of paper folded along a diagonal, creating an M-shaped region.]

Originally question 7 of the 2016 University of North Colorado Math Contest (First Round).

We received eight correct solutions. We present the solution of Doddy Kastanya.

The area of the folded configuration is basically the area of the rectangle minus the area of triangle ACE since there is an overlap there. So, determining the area of the original rectangle is straightforward. The area of the rectangle is 12 × 18 or 216. The base of triangle ACE (line AC) is simply the hypotenuse of triangle ABC (since ∠ABC is right). So, the length of AC is $\sqrt{12^2 + 18^2}$ or $6\sqrt{13}$. The height of triangle ACE is EF which can be determined as $FC \times \tan \angle BAC$. The length of FC is $3\sqrt{13}$. Using triangle ABC, tan ∠BAC is equal to $\frac{12}{18} = \frac{2}{3}$. So, the area of triangle ACE is

$$\frac{1}{2} AC \times EF = \frac{1}{2} (6\sqrt{13}) \times \left(\frac{2}{3} \times 3\sqrt{13}\right) = 78.$$ 

So, the overall shaded area is 216 − 78 or 138.

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CC212. A cube that is one inch wide has had its eight corners shaved off. The cube’s vertices have been replaced by eight congruent equilateral triangles, and the square faces have been replaced by six congruent octagons. If the combined area of the eight triangles equals the area of one of the octagons, what is that area? (Each octagonal face has two different edge lengths that occur in alternating order.)

originally question 3 of the 2016 university of north colorado math contest (final round).

we received three correct and complete solutions, out of which we present the one by john g. heuver.

let the edges of the base of a shaved off tetrahedron be $x$. then the remaining faces of the tetrahedron are right angled isosceles triangles. let the length of the legs of those triangles be $p$, so $2p^2 = x^2$. thus the area of one of the octagons is

$$1 - 4 \cdot \frac{p^2}{2} = 1 - x^2$$

and the combined area of the eight triangles is

$$8 \cdot \frac{\sqrt{3}}{4} x^2 = 2\sqrt{3} x^2.$$

hence $1 - x^2 = 2\sqrt{3} x^2$ or $x^2 = \frac{1}{1 + 2\sqrt{3}}$. substituting this into (1) and rationalizing gives us that the area of one of the octagons in square inches is

$$\frac{12 - 2\sqrt{3}}{11}.$$

cc213. a pyramid is built from solid unit cubes that are stacked in square layers. the top layer has $1 \times 1 = 1$ cube, the second $3 \times 3 = 9$ cubes and the layer below that has $5 \times 5 = 25$ cubes, and so on, with each layer having two more cubes on a side than the layer above it. the pyramid has a total of 12 layers. find the exposed surface area of this solid pyramid, including the bottom.

originally question 8 of the 2016 university of north colorado math contest (first round).

we received four correct and complete solutions. we present the solution of carlos vega and angel plaza.

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Since each layer of the pyramid has two more cubes on a side than the layer above it, the $n$-th layer from the top contains $(2n - 1) \times (2n - 1)$ cubes. Therefore the bottom layer consists of $23 \times 23$ cubes. Thus from below and above one can see $23^2$ squares. From each of the four sides, the number of squares one can see is

$$\sum_{k=1}^{12} (2n - 1) = 12^2.$$ 

Hence the exposed surface area is $2 \times 23^2 + 4 \times 12^2 = 1634$ square units.

**CC214.** The points $(2, 5)$ and $(6, 5)$ are two of the vertices of a regular hexagon of side length two on a coordinate plane. There is a line $L$ that goes through the point $(0, 0)$ and cuts the hexagon into two pieces of equal area. What is the slope of line $L$?

*Originally question 6 of the 2016 University of North Colorado Math Contest (First Round).*

*We received seven submissions of which six were correct and complete. We present the solution by Fernando Ballesta Yagüe.*

We have that $(2, 5), (6, 5)$ are two of the vertices of the hexagon. The segment $AB$ has length 4. In a regular hexagon with side length 2, two vertices whose distance is 4 are opposed to the center. Therefore, the center of the hexagon will be the midpoint of $A$ and $B$, which is $(4, 5)$. Since a regular hexagon is a symmetric figure, a line that divides it in two pieces of equal area will pass through its center. As we know two points of this line (the center and the origin of coordinates), we can find out its slope: $m = \frac{5 - 0}{4 - 0} = \frac{5}{4}$.

**CC215.** Each circle in this tree diagram is to be assigned a value, chosen from a set $S$, in such a way that along every pathway down the tree the assigned values never increase. That is, $A \geq B, A \geq C, C \geq D, C \geq E$ and $A, B, C, D, E \in S$. (It is permissible for a value in $S$ to appear more than once.) How many ways can the tree be so numbered using only values chosen from the set $S = \{1, \ldots, 6\}$?

(Optional extension: Generalize to a case with $S = \{1, 2, 3, \ldots, n\}$ by finding an explicit algebraic expression for the number of ways the tree can be numbered.)
Originally question 8 of the 2016 University of North Colorado Math Contest (Final Round).

We received two correct solutions and one incomplete submission. We present the solution by Steven Chow.

A is any integer between 1 and \( n \). If \( A \) is fixed, then \( B \) and \( C \) are any integers between 1 and \( A \), which are \( A \) possibilities. If \( C \) is also fixed, then there are \( C \) possibilities for each of \( D \) and \( E \). We can thus calculate the number of ways that the tree can be numbered as follows.

\[
\sum_{A=1}^{n} A \sum_{C=1}^{A} C^2 = \sum_{A=1}^{n} A \left( \frac{1}{6} A + \frac{1}{2} A^2 + \frac{1}{3} A^3 \right)
\]

\[
= \frac{1}{6} \sum_{A=1}^{n} A^2 + \frac{1}{2} \sum_{A=1}^{n} A^3 + \frac{1}{3} \sum_{A=1}^{n} A^4
\]

\[
= \frac{1}{6} \left[ \binom{n}{1} + 3 \binom{n}{2} + 2 \binom{n}{3} \right]
\]

\[
+ \frac{1}{2} \left[ \binom{n}{1} + 7 \binom{n}{2} + 12 \binom{n}{3} + 6 \binom{n}{4} \right]
\]

\[
+ \frac{1}{3} \left[ \binom{n}{1} + 15 \binom{n}{2} + 50 \binom{n}{3} + 60 \binom{n}{4} + 24 \binom{n}{5} \right]
\]

\[
= \binom{n}{1} + 9 \binom{n}{2} + 23 \binom{n}{3} + 23 \binom{n}{4} + 8 \binom{n}{5}.
\]

For the special case of \( n = 6 \), this comes to 994 ways to fill the tree.

**Editor's comments.** The formula for the sum of the \( k \)-th powers used in the calculation is

\[
\sum_{k=0}^{n} k^m = \sum_{j=1}^{n} (j - 1)! \left\{ \frac{m + 1}{j} \right\} \binom{n}{j},
\]

where \( \left\{ \frac{m+1}{j} \right\} \) is a Stirling number of the second kind.