CONTEST CORNER
SOLUTIONS


CC211. A rectangular sheet of paper whose dimensions are $12 \times 18$ is folded along a diagonal, which creates the $M$-shaped region drawn at the right. Find the area of the shaded region.

Originally question 7 of the 2016 University of North Colorado Math Contest (First Round).

We received eight correct solutions. We present the solution of Doddy Kastanya.

The area of the folded configuration is basically the area of the rectangle minus the area of triangle $ACE$ since there is an overlap there. So, determining the area of the original rectangle is straight-forward. The area of the rectangle is $12 \times 18$ or 216. The base of triangle $ACE$ (line $AC$) is simply the hypotenuse of triangle $ABC$ (since $\angle ABC$ is right). So, the length of $AC$ is $\sqrt{12^2 + 18^2}$ or $6\sqrt{13}$. The height of triangle $ACE$ is $EF$ which can be determined as $FC \times \tan \angle BAC$. The length of $FC$ is $3\sqrt{13}$. Using triangle $ABC$, $\tan \angle BAC$ is equal to $\frac{12}{18} = \frac{2}{3}$. So, the area of triangle $ACE$ is

$$\frac{1}{2} AC \times EF = \frac{1}{2} (6\sqrt{13}) \times \left(\frac{2}{3} \cdot 3\sqrt{13}\right) = 78.$$

So, the overall shaded area is $216 - 78$ or 138.

Crux Mathematicorum, Vol. 43(3), March 2017
CC212. A cube that is one inch wide has had its eight corners shaved off. The cube’s vertices have been replaced by eight congruent equilateral triangles, and the square faces have been replaced by six congruent octagons. If the combined area of the eight triangles equals the area of one of the octagons, what is that area? (Each octagonal face has two different edge lengths that occur in alternating order.)

Originally question 3 of the 2016 University of North Colorado Math Contest (Final Round).

We received three correct and complete solutions, out of which we present the one by John G. Heuver.

Let the edges of the base of a shaved off tetrahedron be $x$. Then the remaining faces of the tetrahedron are right angled isosceles triangles. Let the length of the legs of those triangles be $p$, so $2p^2 = x^2$. Thus the area of one of the octagons is

$$1 - 4 \cdot \frac{p^2}{2} = 1 - x^2$$

and the combined area of the eight triangles is

$$8 \cdot \frac{\sqrt{3}}{4} x^2 = 2\sqrt{3} x^2.$$

Hence $1 - x^2 = 2\sqrt{3} x^2$ or $x^2 = \frac{1}{1 + 2\sqrt{3}}$. Substituting this into (1) and rationalizing gives us that the area of one of the octagons in square inches is

$$\frac{12 - 2\sqrt{3}}{11}.$$

CC213. A pyramid is built from solid unit cubes that are stacked in square layers. The top layer has $1 \times 1 = 1$ cube, the second $3 \times 3 = 9$ cubes and the layer below that has $5 \times 5 = 25$ cubes, and so on, with each layer having two more cubes on a side than the layer above it. The pyramid has a total of 12 layers. Find the exposed surface area of this solid pyramid, including the bottom.

Originally question 8 of the 2016 University of North Colorado Math Contest (First Round).

We received four correct and complete solutions. We present the solution of Carlos Vega and Angel Plaza.
Since each layer of the pyramid has two more cubes on a side than the layer above it, the $n$-th layer from the top contains $(2n - 1) \times (2n - 1)$ cubes. Therefore the bottom layer consists of $23 \times 23$ cubes. Thus from below and above one can see $23^2$ squares. From each of the four sides, the number of squares one can see is

\[
\sum_{k=1}^{12} (2n - 1) = 12^2.
\]

Hence the exposed surface area is $2 \times 23^2 + 4 \times 12^2 = 1634$ square units.

**CC214.** The points $(2, 5)$ and $(6, 5)$ are two of the vertices of a regular hexagon of side length two on a coordinate plane. There is a line $L$ that goes through the point $(0, 0)$ and cuts the hexagon into two pieces of equal area. What is the slope of line $L$?

*Originally question 6 of the 2016 University of North Colorado Math Contest (First Round).*

*We received seven submissions of which six were correct and complete. We present the solution by Fernando Ballesta Yagüe.*

We have that $(2, 5), (6, 5)$ are two of the vertices of the hexagon. The segment $AB$ has length 4. In a regular hexagon with side length 2, two vertices whose distance is 4 are opposed to the center. Therefore, the center of the hexagon will be the midpoint of $A$ and $B$, which is $(4, 5)$. Since a regular hexagon is a symmetric figure, a line that divides it in two pieces of equal area will pass through its center. As we know two points of this line (the center and the origin of coordinates), we can find out its slope: $m = \frac{5 - 0}{4 - 0} = \frac{5}{4}$.

**CC215.** Each circle in this tree diagram is to be assigned a value, chosen from a set $S$, in such a way that along every pathway down the tree the assigned values never increase. That is, $A \geq B, A \geq C, C \geq D, C \geq E$ and $A, B, C, D, E \in S$. (It is permissible for a value in $S$ to appear more than once.) How many ways can the tree be so numbered using only values chosen from the set $S = \{1, \ldots, 6\}$?

(Optional extension: Generalize to a case with $S = \{1, 2, 3, \ldots, n\}$ by finding an explicit algebraic expression for the number of ways the tree can be numbered.)

*Crux Mathematicorum, Vol. 43(3), March 2017*
Originally question 8 of the 2016 University of North Colorado Math Contest (Final Round).

We received two correct solutions and one incomplete submission. We present the solution by Steven Chow.

A is any integer between 1 and n. If A is fixed, then B and C are any integers between 1 and A, which are A possibilities. If C is also fixed, then there are C possibilities for each of D and E. We can thus calculate the number of ways that the tree can be numbered as follows.

\[
\sum_{A=1}^{n} A \sum_{C=1}^{A} C^2 = \sum_{A=1}^{n} A \left( \frac{1}{6} A + \frac{1}{2} A^2 + \frac{1}{3} A^3 \right)
\]

\[
= \frac{1}{6} \sum_{A=1}^{n} A^2 + \frac{1}{2} \sum_{A=1}^{n} A^3 + \frac{1}{3} \sum_{A=1}^{n} A^4
\]

\[
= \frac{1}{6} \left[ \binom{n}{1} + 3 \binom{n}{2} + 2 \binom{n}{3} \right]
\]

\[
+ \frac{1}{2} \left[ \binom{n}{1} + 7 \binom{n}{2} + 12 \binom{n}{3} + 6 \binom{n}{4} \right]
\]

\[
+ \frac{1}{3} \left[ \binom{n}{1} + 15 \binom{n}{2} + 50 \binom{n}{3} + 60 \binom{n}{4} + 24 \binom{n}{5} \right]
\]

\[
= \binom{n}{1} + 9 \binom{n}{2} + 23 \binom{n}{3} + 23 \binom{n}{4} + 8 \binom{n}{5}.
\]

For the special case of n = 6, this comes to 994 ways to fill the tree.

Editor’s comments. The formula for the sum of the k-th powers used in the calculation is

\[
\sum_{k=0}^{n} k^m = \sum_{j=1}^{n} (j-1)! \binom{m+1}{j} \binom{n}{j},
\]

where \( \binom{m+1}{j} \) is a Stirling number of the second kind.