intervals \{m + 1, m + 2, \ldots, m + A\} consisting of \(A\) consecutive positive integers, and places lava at all of the integers in the intervals. The intervals must be chosen so that:

1. any two distinct intervals are disjoint and not adjacent;
2. there are at least \(F\) positive integers with no lava between any two intervals; and
3. no lava is placed at any integer less than \(F\).

Prove that the smallest \(F\) for which the flea can jump over all the intervals and avoid all the lava, regardless of what Lavaman does, is \(F = (n - 1)A + B\), where \(n\) is the positive integer such that \(\frac{A}{n+1} \leq B - A < \frac{A}{n}\).

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**OLYMPIAD SOLUTIONS**


**OC296.** Let \(\mathbb{N} = \{1, 2, 3, \ldots\}\) be the set of positive integers. Find all functions \(f\), defined on \(\mathbb{N}\) and taking values in \(\mathbb{N}\), such that \((n - 1)^2 < f(n)f(f(n)) < n^2 + n\) for every positive integer \(n\).

*Originally problem 1 of the 2015 Canadian Mathematical Olympiad.*

*We received 4 correct submissions. We present the solution by Joseph Ling.*

It is easy to see that the function \(f(n) = n\) satisfies the inequalities
\[(n - 1)^2 < f(n)f(f(n)) < n^2 + n\]
for every positive integer \(n\). We claim that it is the only function with this property.

Suppose that for some value of \(n\), \(f(f(n)) \leq f(n)\). Then we have both
\[(n - 1)^2 < f(n)f(f(n)) \leq f(n)^2 \implies n - 1 < f(n) \implies n \leq f(n)\]
and
\[f(f(n))^2 \leq f(n)f(f(n)) < n^2 + n < (n + 1)^2 \implies f(f(n)) < n + 1 \implies f(f(n)) \leq n.\]
This shows that \( n \) lies (weakly) between \( f(n) \) and \( f(f(n)) \). Similarly, \( n \) must lie (weakly) between \( f(n) \) and \( f(f(n)) \) also in the case where \( f(n) \leq f(f(n)) \).

Now, for any positive integer \( n \), consider the sequence \( \{a_k\}_{k=1}^\infty \) defined by \( a_0 = n \) and \( a_{k+1} = f(a_k) \) for all non-negative integers \( k \). Consider the case where \( a_0 \leq a_1 \). The argument for the case with \( a_0 > a_1 \) is similar. By the above analysis, we see that \( a_k \) always lies between \( a_{k+1} \) and \( a_{k+2} \). Therefore, an inductive argument yields the following relative ordering of the terms of the sequence:

\[ \cdots \leq a_{2k+2} \leq a_{2k} \leq \cdots \leq a_2 \leq a_0 \leq a_1 \leq \cdots \leq a_{2k-1} \leq a_{2k+1} \leq \cdots \]

By the well-ordering principle, the even-index subsequence \( \{a_{2k}\}_{k=1}^\infty \) must eventually take a constant value, say \( L_o \). It follows that the odd-index subsequence \( \{a_{2k-1}\}_{k=1}^\infty \) is also eventually a constant, say \( L_e \), where \( L_o \leq L_e \). As well, \( f(L_o) = L_e \) and \( f(L_e) = L_o \). It suffices to show that \( L_o = L_e \). For this will imply that \( \{a_k\}_{k=1}^\infty \) is a constant sequence, and in particular, \( f(n) = n \).

Suppose the contrary, i.e., suppose that \( L_o < L_e \). Since

\[ L_o L_e = f(L_o) f(f(L_o)) = f(L_e) f(f(L_e)) , \]

this common value satisfies both

\[ (L_o - 1)^2 < L_o L_e < L_o^2 + L_o \quad \text{and} \quad (L_e - 1)^2 < L_o L_e < L_e^2 + L_e . \]

Since \( L_o < L_e \), this can happen only if \( (L_e - 1)^2 < L_o^2 + L_o \), which implies that \( L_o - 1 < L_o + 1 \). It follows that \( L_e = L_o + 1 \). But then, \( L_o L_e = L_o^2 + L_o \), a contradiction. Our proof is complete.

**OC297.** Prove that \( \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n+1)^2} < n \cdot \left(1 - \frac{1}{\sqrt{2}} \right) . \)

*Originally problem 1 from day 1 of the 2015 Kazakhstan National Olympiad.*

We received 6 correct submissions. We present the solution by Michel Bataille.

The inequality rewrites as \( X > n/\sqrt{2} \), where

\[ X = \left(1 - \frac{1}{2^2}\right) + \left(1 - \frac{1}{3^2}\right) + \cdots + \left(1 - \frac{1}{(n+1)^2}\right) = \frac{1}{2^2} + \frac{2 \cdot 4}{3^2} + \cdots + \frac{n(n+2)}{(n+1)^2} . \]

Now, by AM-GM, we obtain \( X > Y \) where

\[
Y = n \cdot \sqrt{\frac{1 \cdot 3}{2^2} \cdot \frac{2 \cdot 4}{3^2} \cdots \frac{n(n+2)}{(n+1)^2}}
= n \cdot \sqrt{(n!) \cdot \frac{2}{(2(n+1)!)} \cdot \frac{1}{(n!)^2}}
= \frac{n}{\sqrt{2}} \left( \frac{n(n+2)!}{(n+1)! (n+1)!} \right)^{1/n}
= \frac{n}{\sqrt{2}} \left( \frac{n+2}{n+1} \right)^{1/n} .
\]
Since \( \frac{n+2}{n+1} > 1 \), we have \( \left( \frac{n+2}{n+1} \right)^{1/n} > 1 \), hence \( Y > \frac{n}{\sqrt[4]{2}} \). Thus, \( X > Y > \frac{n}{\sqrt[4]{2}} \) and the desired inequality \( X > Y > \frac{n}{\sqrt[4]{2}} \) follows.

**OC298.** Triangle \( ABC \) is an acute triangle and its orthocenter is \( H \). The circumcircle of \( \triangle ABH \) intersects line \( BC \) at \( D \). Lines \( DH \) and \( AC \) meet at \( P \), and the circumcenter of \( \triangle ADP \) is \( Q \). Prove that the circumcenter of \( \triangle ABH \) lies on the circumcircle of \( \triangle BDQ \).

*Originally problem 1 from day 2 of the 2015 Final Round of the Korean National Olympiad.*

We received 4 correct submissions. We present the solution by Steven Chow.

Directed angles are used (mod \( \pi \)). Let \( B = \angle CBA \) and \( C = \angle ACB \) for short. Let \( E \) be the circumcenter of \( \triangle ABH \).

Since \( A, B, D, \) and \( H \) are concyclic and \( H \) is the orthocentre of acute \( \triangle ABC \), \( \angle BDA = \angle BHA = C \).

Since \( Q \) is the circumcenter of \( \triangle ADP \),

\[
\angle QAC = \angle QAP = \frac{1}{2} \pi - \angle PDA = \frac{1}{2} \pi - \angle HDA = \frac{1}{2} \pi - \angle HBA = \angle BAC,
\]

so \( Q \) is on \( \overrightarrow{AB} \).

Therefore

\[
\angle QDA = \angle DAQ = \angle DAB = \angle CBA - \angle BDA = B - C,
\]

so

\[
\angle BDQ = \angle BDA - \angle QDA = C - (B - C) = -B + 2C.
\]

Since \( E \) is the circumcenter of \( \triangle ABD \),

\[
\angle QBE = \angle ABE = \frac{1}{2} \pi - \angle BDA = \frac{1}{2} \pi - C.
\]

The radical axis of circles \( (ADP) \) and \( (ABDH) \) is \( \overrightarrow{AD} \), so \( \overrightarrow{AD} \perp \overrightarrow{EQ} \), so

\[
\angle EQB = \frac{1}{2} \pi + \angle DAQ = \frac{1}{2} \pi + B - C.
\]

Therefore \( \angle BEQ = \pi - \angle QBE - \angle EQB = -B + 2C = \angle BDQ \), so \( B, D, E, \) and \( Q \) are concyclic.

Therefore the circumcenter of \( \triangle ABH \) lies on the circumcircle of \( \triangle BDQ \).

*Crux Mathematicorum, Vol. 43(10), December 2017*
OC299. Find all positive integers $k$ such that for any positive integer $n$, $2^{(k-1)n+1}$ does not divide $\frac{(kn)!}{n!}$.


We present the solution by Mohammed Aassila. There were no other submissions.

We will prove that the set of positive integers $k$ such that for any positive integer $n$, the integer $2^{(k-1)n+1}$ does not divide $\frac{(kn)!}{n!}$ is the set of all powers of 2. By Legendre's Formula, we know that

$$v_p(n!) = \frac{n - S_p(n)}{p - 1}$$

where $S_p(n)$ is the sum of the digits of $n$ when written in base $p$ and $v_p$ is the usual $p$-adic valuation. We have

$$v_2\left(\frac{(kn)!}{n!}\right) = (k - 1)n + (S_2(n) - s_2(kn)) = (k - 1)n + (S_2(n) - s_2(kn)).$$

Thus, the condition of the problem is equivalent to $S_2(kn) \geq S_2(n)$.

If $k = 1$, then this is trivially true. Notice that this is satisfied if and only if $2k$ satisfies the condition as well. Therefore, it suffices to show that there is no such $k > 1$ for which $2^{(k-1)n+1}$ does not divide $\frac{(kn)!}{n!}$ for all $n$.

Let $m$ be a multiple of $k$ with a minimal number of 1 digits (say $v$ many) and take $m$ to be odd. Then, let $m = 2^a + b$ with $2^a > b$. Notice that the number

$$w = 2^a + \varphi(k) + b$$

is also a multiple of $k$ with $v$ ones for every $s \in \mathbb{N}$ where $\varphi(k)$ is Euler’s phi function.

We now prove that $w/k$ must have more than $v$ digits that are 1 for sufficiently large $s$ and so does not satisfy the condition of the question. Assume otherwise and pick $s$ so that $w/k$ has at least $z$ consecutive zeroes where $z$ is the length of the binary representation of $k$. Then for sufficiently large $s$, the number of one digits in the binary representation of $w/k \cdot k$ is greater than $v$, a contradiction.

OC300. All contestants at one contest are sitting in $n$ columns and are forming a “good” configuration. (We define one configuration as “good” when we don’t have 2 friends sitting in the same column). It’s impossible for all the students to sit in $n - 1$ columns in a “good” configuration. Prove that we can always choose contestants $M_1, M_2, \ldots, M_n$ such that $M_i$ is sitting in the $i$th column, for each $i = 1, 2, \ldots, n$ and $M_i$ is friend of $M_{i+1}$ for each $i = 1, 2, \ldots, n - 1$.

Originally problem 3 of the 2015 Macedonia National Olympiad.

We received 2 correct submissions. We present the solution by Oliver Guepel.
Our proof is by contradiction. Suppose that the result is false. We will construct a good configuration consisting of at most $n - 1$ columns, which contradicts the hypothesis of the problem.

Let $\mathcal{C}_i$ denote the set of contestants sitting in the $i$th column, $1 \leq i \leq n$. For contestants $M$ and $N$, we write $M \rightarrow N$ when $M$ and $N$ are friends and $M \in \mathcal{C}_i$, $N \in \mathcal{C}_{i+1}$, for some $i \in \{1, \ldots, n-1\}$. Let us say that contestant $M$ is reachable if there is a chain $M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_i$ such that $M_1 \in \mathcal{C}_1$ and $M_i = M$ for some $i \in \{1, \ldots, n\}$. (The members of $\mathcal{C}_1$ are reachable.) A contestant is called unreachable if he or she is not reachable. By hypothesis, all the students in the $n$th column are unreachable.

Let $\Gamma_1$ denote the given good configuration. We construct a new good configuration $\Gamma_2$ as follows. Choose an unreachable participant $M \in \mathcal{C}_i$ such that the number $i \geq 2$ is minimal. Contestant $M$ has no friends in $\mathcal{C}_{i-1}$ since the members of $\mathcal{C}_{i-1}$ are reachable but $M$ is not. As a consequence, we may shift $M$ to column $\mathcal{C}_{i-1}$ to obtain our desired good configuration $\Gamma_2$. It is crucial to observe that $\Gamma_2$ has not more than $n$ columns and the remaining members of column $\mathcal{C}_n$ remain unreachable in $\Gamma_2$.

We iterate the same procedure as long as an unreachable contestant can be found. The process is finite because a student is shifted to a "lesser" column in each step. After a finite number of steps we arrive at a good configuration where all contestants are reachable. Consequently, column $\mathcal{C}_n$ is empty in this final configuration. The proof is complete.