OC311. Let \( \triangle ABC \) be an acute-scalene triangle, and let \( N \) be the center of the circle which passes through the feet of the altitudes. Let \( D \) be the intersection of the tangents to the circumcircle of \( \triangle ABC \) at \( B \) and \( C \). Prove that \( A, D \) and \( N \) are collinear if and only if \( \angle BAC = 45^\circ \).

OC312. Let \( a, b, c \) be nonnegative real numbers. Prove that
\[
\frac{(a - bc)^2 + (b - ca)^2 + (c - ab)^2}{(a - b)^2 + (b - c)^2 + (c - a)^2} \geq \frac{1}{2}.
\]

OC313. Let \( x_1, x_2, \ldots, x_n \in (0, 1) \), \( n \geq 2 \). Prove that
\[
\frac{\sqrt{1 - x_1}}{x_1} + \frac{\sqrt{1 - x_2}}{x_2} + \cdots + \frac{\sqrt{1 - x_n}}{x_n} < \frac{\sqrt{n - 1}}{x_1 x_2 \cdots x_n}.
\]

OC314. Find all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that for all reals \( x, y, z \), we have
\[
(f(x) + 1)(f(y) + f(z)) = f(xy + z) + f(xz - y).
\]

OC315. Suppose that \( a \) is an integer and that \( n! + a \) divides \( (2n)! \) for infinitely many positive integers \( n \). Prove that \( a = 0 \).

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OC311. Soit \( ABC \) un triangle scalène acutangle et \( N \) le centre du cercle qui passe aux pieds des trois hauteurs du triangle. Soit \( D \) le point d’intersection des tangentes au cercle circonscrit au triangle \( ABC \) aux sommets \( B \) et \( C \). Démontrer que les points \( A, D \) et \( N \) sont alignés si et seulement si \( \angle BAC = 45^\circ \).

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OC312. Soit $a, b$ et $c$ des réels positifs ou nuls. Démontrer que
\[
\frac{(a - bc)^2 + (b - ca)^2 + (c - ab)^2}{(a - b)^2 + (b - c)^2 + (c - a)^2} \geq \frac{1}{2}.
\]

OC313. Soit $x_1, x_2, \cdots, x_n \in (0, 1)$, $n \geq 2$. Démontrer que
\[
\sqrt{1 - x_1} + \sqrt{1 - x_2} + \cdots + \sqrt{1 - x_n} < \frac{\sqrt{n} - 1}{x_1 x_2 \cdots x_n}.
\]

OC314. Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que
\[
(f(x) + 1)(f(y) + f(z)) = f(xy + z) + f(xz - y)
\]
pour tous réels $x, y$ et $z$.

OC315. Soit $a$ un entier tel que $n! + a$ soit un diviseur de $(2n)!$ pour un nombre infini d’entiers strictement positifs $n$. Démontrer que $a = 0$.

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**OLYMPIAD SOLUTIONS**


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OC251. Let $a, b, c, d$ be real numbers such that $b - d \geq 5$ and all zeros $x_1, x_2, x_3$, and $x_4$ of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

*Originally problem 1 from day 1 of the 2014 USAMO.*

We present the solution by Missouri State University Problem Solving Group. There were no other submissions.

We have that $P(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$ and so
\[
\begin{align*}
(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1) & = (i - x_1)(i - x_2)(i - x_3)(i - x_4) \cdot (-i - x_1)(-i - x_2)(-i - x_3)(-i - x_4) \\
& = P(i)P(-i) \\
& = (1 - ai - b + ci + d)(1 + ai - b - ci + d) \\
& = ((1 - b + d) - (a - c)i)((1 - b + d) + (a - c)i) \\
& = (1 - b + d)^2 + (a - c)^2 \\
& = (b - d - 1)^2 + (a - c)^2 \\
& \geq 4^2 + (a - c)^2
\end{align*}
\]

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This is smallest when $a = c$ and in that case, the minimum value is 16. Note that $(x - 1)^4$ shows that this minimum can be obtained.

**OC252.** In an obtuse triangle $ABC$ ($AB > AC$), $O$ is the circumcentre and $D, E, F$ are the midpoints of $BC, CA, AB$ respectively. Median $AD$ intersects $OF$ and $OE$ at $M$ and $N$ respectively. $BM$ meets $CN$ at point $P$. Prove that $OP \perp AP$.

*Originally problem 3 from day 1 of the 2014 South East Mathematical Olympiad.*

We received 2 correct submissions. We present the solution by Andrea Fanchini.

We use barycentric coordinates and the usual Conway’s notations with reference to triangle $ABC$.

Coordinates of points $D, E, F, O$. These points are well known

$D(0 : 1 : 1), \quad E(1 : 0 : 1), \quad F(1 : 1 : 0), \quad O(a^2S_A : b^2S_B : c^2S_C)$.

Equations of lines $AD, OE, OF$. Now the equations of these lines are

$AD : y - z = 0, \quad OE : b^2x + (a^2 - c^2)y - b^2z = 0, \quad OF : c^2x - c^2y + (a^2 - b^2)z = 0$.

Coordinates of points $M$ and $N$. We have

$M = AD \cap OF = (2S_A : c^2 : c^2), \quad N = AD \cap OE = (2S_A : b^2 : b^2)$.

Equations of lines $BM, CN$. Now the equations of these lines are

$BM : c^2x - 2S_Az = 0, \quad CN : b^2x - 2S_Ay = 0$.

Coordinates of point $P$. We have

$P = BM \cap CN = (2S_A : b^2 : c^2)$

Equations of lines $AP, OP$. Now the equations of these lines are

$AP : c^2y - b^2z = 0, \quad OP : b^2x - c^2S_Ay - b^2S_Az = 0$.

Perpendicularity of $AP$ and $OP$. The infinite perpendicular point of line $AP$ is

$AP_{\infty} \left( S_A(b^2 - c^2) : -b^2(S_A + c^2) : c^2(S_A + b^2) \right)$.

then the infinite point of $OP$ is

$OP_{\infty} \left( S_A(b^2 - c^2) : -b^2(S_A + c^2) : c^2(S_A + b^2) \right)$.

so $AP_{\infty, \perp} = OP_{\infty}$, therefore $AP$ and $OP$ are perpendicular.

**OC253.** Prove that there exist infinitely many positive integers $n$ such that $3^n + 2$ and $5^n + 2$ are all composite numbers.

*Originally problem 8 of the 2014 China Northern Mathematical Olympiad.*

We received 9 correct submissions. We present the solution by Ali Adnan.

It is easily seen using Fermat’s Little Theorem that for all $k \geq 0$,
which shows that there are infinitely many $n \in \mathbb{N}$ such that both $3^n + 2$ and $5^n + 2$ are composite ($n = 16k + 6, \ k \geq 0$).

**OC254.** Find all non-negative integers $k, n$ which satisfy $2^{2k+1} + 9 \cdot 2^k + 5 = n^2$.

*Originally problem 5 of the 2014 Balkan Mathematical Olympiad Team Selection Test.*

*We received 9 correct submissions. We present the solution by Titu Zvonaru.*

The only solution is when $k = 0$ and $n = 4$ which one can easily check is a solution. By inspection, we can plug in the values for $k = 0, \ k = 1,$ and $k = 2$ and see that the only solution arises when $k = 0$ which we have already considered. Now, suppose that $k > 2$. Since the left hand side is odd, it follows that $n$ is odd. Let $n = 2p + 1$ where $p$ is a positive integer. The given equation is equivalent to

$$2^{2k+1} + 9 \cdot 2^k + 5 = 4p^2 + 4p + 1$$

$$2^{2k-1} + 9 \cdot 2^{k-2} + 5 = p(p + 1)$$

As $k > 2$, the left hand side above is odd however the right hand side above is even and thus we do not obtain any solutions.

**OC255.** Let $n$ be a positive integer and $x_1, x_2, \ldots, x_n$ be positive reals. Show that there are numbers $a_1, a_2, \ldots, a_n \in \{-1, 1\}$ such that the following holds:

$$a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2 \geq (a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)^2$$

*Originally problem 6 of the 2014 France Team Selection Test.*

*We received 1 incorrect submission.*

*Editor’s Note:* There is a requirement that $|a_i| = 1$ for each $i$. 

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