SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


4071. Proposed by Leonard Giugiuc and Daniel Sitaru.

Prove that if \(a, b, c \in (0, 1)\), then \(a^{a+1}b^{b+1}c^{c+1} < e^{2(a+b+c)-6}\).

There were 14 correct solutions and one incorrect submission. We present the solution submitted by various solvers.

Let

\[ f(x) = 2(x - 1) - (x + 1) \ln x = 2(x - 1) - x \ln x - \ln x \]

for \(0 < x < 1\). The derivative

\[ f'(x) = 1 - x^{-1} - \ln x = \ln(x^{-1}) - (x^{-1} - 1) \]

is negative and \(f(1) = 0\), so that \(f(x) > 0\) on \((0, 1)\). Therefore \((x+1) \ln x < 2(x-1)\) for \(0 < x < 1\).

Thus

\[ (a + 1) \ln a + (b + 1) \ln b + (c + 1) \ln c < 2(a + b + c) - 6. \]

Exponentiating yields the result.

Editor’s Comments. Several solvers used the function in the solution; another popular function studied was \(\ln x - 2(x-1)(x+1)^{-1} = \ln x + 4(x+1)^{-1} - 2\) or a close relative. Three solvers noted that the function \((x+1) \ln x\) was concave and applied Jensen’s inequality. One respondent took a stroll down the garden path with the following argument.

Let \(L = a^{a+1}b^{b+1}c^{c+1}\). Since \(\ln x < x - 1\) for \(0 < x < 1\),

\[ \ln L = (a + 1) \ln a + (b + 1) \ln b + (c + 1) \ln c < 2(\ln a + \ln b + \ln c) \]

\[ < 2((a - 1) + (b - 1) + (c - 1)) = 2(a + b + c) - 6. \]

Exponentiating yields the desired result.

4072. Proposed by Michel Bataille.

Let \(a, b\) be distinct positive real numbers and \(A = \frac{a+b}{2}, G = \sqrt{ab}, L = \frac{a-b}{\ln a - \ln b}\).

Prove that

\[ \frac{L}{G} > \frac{4A + 5G}{A + 8G}. \]

There were five correct solutions, of which we present two.

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Solution 1, by the proposer.

This inequality is a refinement of the known inequality $L > G$. Wolog, let $a > b$ and set $x = \sqrt{a}$, $y = \sqrt{b}$. The inequality can be rewritten as

$$\frac{\ln x - \ln y}{x^2 - y^2} < \frac{1}{4xy} \cdot \frac{x^2 + y^2 + 16xy}{2x^2 + 2y^2 + 5xy}.$$  

Setting $t = x/y$ converts it to

$$\frac{4\ln t}{t^2 - 1} < \frac{1}{t} \cdot \frac{t^2 + 16t + 1}{2t^2 + 5t + 2}.$$  

Since

$$\frac{1}{t} \cdot \frac{t^2 + 16t + 1}{2t^2 + 5t + 2} = \frac{1}{2} \left( \frac{1}{t} + \frac{27}{2t^2 + 5t + 2} \right),$$

it all boils down to proving for $t > 1$ the inequality

$$\frac{8\ln t}{t^2 - 1} < \frac{1}{t} + \frac{27}{2t^2 + 5t + 2}.$$  

Recall Simpson’s $\frac{3}{8}$ Rule for numerical integration of a $C^4$-function $f$ on a closed interval $[u, v]$

$$\int_u^v f(s)ds = \frac{v - u}{8} \left( f(u) + 3f\left( \frac{2u + v}{3} \right) + 3f\left( \frac{u + 2v}{3} \right) + f(v) \right) - \frac{(v - u)^5}{6480} f^{(4)}(\xi)$$

for some $\xi \in (u, v)$. Taking $f(s) = 1/s$, $u = 1$ and $v = t > 1$, we obtain

$$\ln t = \frac{t - 1}{8} \left(1 + \frac{9}{2 + t} + \frac{9}{1 + 2t} + \frac{1}{t} \right) - \frac{24(t - 1)^5}{6480\xi^2}$$

and so

$$\ln t < \frac{t - 1}{8} \left(1 + \frac{1}{t} + \frac{27(t + 1)}{2t^2 + 5t + 2} \right) = \frac{t^2 - 1}{8} \left(1 + \frac{27}{2t^2 + 5t + 2} \right).$$

This leads to the desired inequality.

Solution 2, by Arkady Alt.

Assume $a > b$ and let $t = \sqrt{a/b}$. Then as in the previous solution, we have to establish that

$$4\ln t < \frac{(t^2 - 1)(t^2 + 16t + 1)}{t(t + 2)(2t + 1)}.$$  

Let

$$h(t) = \frac{(t^2 - 1)(t^2 + 16t + 1)}{t(t + 2)(2t + 1)} - 4\ln t$$

$$= \frac{t}{2} - \frac{27}{2(t + 2)} - \frac{27}{4(2t + 1)} - \frac{1}{2t} + \frac{27}{4} - 4\ln t.$$
Then
\[ h'(t) = \frac{1}{2} + \frac{27}{2(t+2)^2} + \frac{27}{2(2t+1)^2} + \frac{1}{2t^2} - \frac{4}{t} \]
\[ = \frac{2(t^2 + t + 1)(t - 1)^4}{t^2(t + 2)^2(2t + 1)^2} > 0 \]
for \( t > 1 \). Since \( h(t) > h(1) = 0 \) for \( t > 1 \), the inequality follows.

4073. Proposed by Daniel Sitaru.

Solve the following system:
\[
\begin{align*}
\sin 2x + \cos 3y &= -1, \\
\sqrt{\sin^2 x + \sin^2 y} + \sqrt{\cos^2 x + \cos^2 y} &= 1 + \sin(x + y).
\end{align*}
\]

*The solution from Michel Bataille was the only one of the 2 submissions that was complete and correct. We present his solution.*

We first show that the second equation is equivalent to \( x + y \equiv \frac{\pi}{2} \) (mod 2\( \pi \)).

If \( x + y \equiv \frac{\pi}{2} \) (mod 2\( \pi \)), then \( \sin^2 y = \cos^2 x \) and \( \cos^2 y = \sin^2 x \). It immediately follows that both sides of the equation equal 2. Conversely, if the equation holds, then squaring gives

\[
2\sqrt{(\sin x \cos x - \sin y \cos y)^2 + \sin^2(x + y)} = 2\sin(x + y) - (1 - \sin^2(x + y)),
\]
and therefore

\[
2\sin(x + y) \leq 2\sqrt{\sin^2(x + y)} \leq 2\sqrt{(\sin x \cos x - \sin y \cos y)^2 + \sin^2(x + y)} = 2\sin(x + y) - (1 - \sin^2(x + y)) \leq 2\sin(x + y).
\]

Thus, equality must hold throughout and in particular \( \sin(x + y) \geq 0 \) and \( \sin^2(x + y) = 1 \). We deduce that \( x + y \equiv \frac{\pi}{2} \) (mod 2\( \pi \)).

Since \( \cos 3\left(\frac{\pi}{2} - x\right) = -\sin 3x \), we are led to seek the solutions to the equation \( f(x) = 1 \) where \( f(x) = \sin 3x - \sin 2x \). Note that \( f\left(-\frac{\pi}{2}\right) = 1 \) so that the numbers \(-\frac{\pi}{2} + 2k\pi \ (k \in \mathbb{Z}) \) are solutions. For other solutions note that \( f \) is odd and 2\( \pi \)-periodic; consequently, we may restrict the study of \( f \) to the interval \([0, \pi]\) and look for \( x \) satisfying either \( f(x) = 1 \) or \( f(x) = -1 \) (the latter since then \( f(-x) = 1 \)). Consider first the interval \([0, \frac{\pi}{2}]\). We have \( f(0) = 0 \) and if \( x \in (0, \frac{\pi}{2}) \), then \( \sin 2x > 0 \) and so \( f(x) < 1 \).

- \( x \in (0, \frac{\pi}{3}] : \sin 3x > 0 \) for \( x \) between 0 and \( \frac{\pi}{3} \), hence \( f(x) > 1 \); since \( f\left(\frac{\pi}{3}\right) > -1 \), there is no \( x \in (0, \frac{\pi}{3}] \) such that \( f(x) = -1 \).

- \( x \in (\frac{\pi}{3}, \frac{\pi}{2}) : f''(x) = 4\sin 2x - 9\sin 3x > 0 \), hence \( f'(x) = 3\cos 3x - 2\cos 2x \) is nondecreasing on the interval \((\frac{\pi}{3}, \frac{\pi}{2})\). For some \( x_1 \in (\frac{\pi}{3}, \frac{\pi}{2}) \), we have \( f'(x) \leq 0 \).

*Crux Mathematicorum, Vol. 42(8), October 2016*
for \( x \in \left( \frac{\pi}{3}, x_1 \right] \) and \( f'(x) > 0 \) for \( x \in (x_1, \frac{\pi}{2}) \). Since \( f\left( \frac{\pi}{3} \right) = -\frac{\sqrt{3}}{2} \) and \( f\left( \frac{\pi}{2} \right) = -1 \), we have \( f(x_1) < -1 \) and \( f(\alpha) = -1 \) for a unique \( \alpha \in \left( \frac{\pi}{3}, \frac{\pi}{2} \right) \).

In a similar way we treat the interval \( (\frac{\pi}{2}, \pi] \). We have \( f(\pi) = 0 \) and if \( x \in (\frac{\pi}{2}, \pi) \), then \( \sin 2x < 0 \), hence \( f(x) > -1 \).

- \( x \in \left( \frac{\pi}{3}, \frac{3\pi}{4} \right) : f'(x) > 0 \) and so \( f \) is increasing from \(-1\) to \( 1 + \frac{\sqrt{3}}{2} \). Thus, \( f(\beta) = 1 \) for a unique \( \beta \in \left( \frac{\pi}{3}, \frac{3\pi}{4} \right) \).

- \( x \in \left( \frac{3\pi}{4}, \pi \right) : f \) is decreasing from \( 1 + \frac{\sqrt{3}}{2} \) to \( 0 \), hence \( f(\gamma) = 1 \) for a unique \( \gamma \) of \( \left( \frac{3\pi}{4}, \pi \right) \).

- \( x \in \left[ \frac{3\pi}{4}, \frac{5\pi}{6} \right] : \) Resorting to \( f''(x) \), we see that \( f'(x) \) decreases from positive to negative so that \( f(x) > 1 \).

In conclusion, on the interval \([-\pi, \pi]\) the solutions \((x, y)\) of the system are the pairs \( (-\frac{\pi}{2}, \pi), (-\alpha, \frac{\pi}{2} + \alpha), (\beta, \frac{\pi}{2} - \beta), \) and \( (\gamma, \frac{\pi}{2} - \gamma) \).

All other solutions are obtained by adding multiples of \( 2\pi \) to \( x \) or \( y \).

4074. Proposed by Abdilkadir Altınış.

Consider the triangle \( ABC \) with the following measures:

Show that \( a + b = c \); that is, \(|AE| + |AC| = |AB|\).

We received 16 solutions, all correct. We feature the solution of C.R. Pranesachar that is typical of the approach used by most solvers.

Because the angles of \( \triangle BCE \) sum to \( 180^\circ \), we see that \( \angle BED = 100^\circ \) and its supplement \( \angle BEA = 80^\circ \). It follows that in \( \triangle BEA \) we also have \( \angle BAE = 80^\circ \), so that \( BE = AB = c \) and

\[
\frac{AE/2}{AB} = \sin \frac{20^\circ}{2},
\]

or

\[
a = 2c \sin 10^\circ.
\]
Further, by the Sine Rule applied to triangle $BCE$,

$$CE = BE \frac{\sin 10^\circ}{\sin 30^\circ} = 2c \sin 10^\circ = a.$$ 

This means that $\triangle AEC$ is also isosceles (with $EA = EC = a$), and because its exterior angle at $E$ equals $40^\circ$, its interior angles at $A$ and $C$ must each be $20^\circ$; thus

$$\frac{b}{2a} = \cos 20^\circ \quad \text{or} \quad b = 2(2c \sin 10^\circ) \cdot \cos 20^\circ.$$ 

The given relation $a + b = c$ now translates to

$$2 \sin 10^\circ + 4 \sin 10^\circ \cos 20^\circ = 1.$$ 

This is easy to prove:

$$\text{lhs} = 2 \sin 10^\circ + 2(\sin 30^\circ - \sin 10^\circ) = 1 = \text{rhs}.$$

4075. Proposed by Leonard Giugiuc and Daniel Sitaru.

Prove that in any triangle $ABC$ with $BC = a$, $CA = b$, $AB = c$ the following inequality holds:

$$\sqrt[3]{abc} \cdot \sqrt{a^2 + b^2 + c^2} \geq 4[ABC],$$

where $[ABC]$ is the area of triangle $ABC$.

We received 16 correct solutions and we present the solution by Martin Lukarevski.

The inequality can be sharpened to

$$\sqrt[3]{abc} \cdot \sqrt{ab + bc + ca} \geq 4[ABC].$$

We use the inequality

$$\sqrt[3]{abc} \geq \sqrt[4]{4[ABC]/\sqrt{3}},$$

which is equivalent to the well-known inequality

$$\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8},$$


$$\sqrt{ab + bc + ca} \geq \sqrt{4[ABC]/\sqrt{3}}.$$ 

Hence

$$\sqrt[3]{abc} \cdot \sqrt{a^2 + b^2 + c^2} \geq \sqrt[3]{abc} \cdot \sqrt{ab + bc + ca} \geq 4[ABC].$$

Editor’s Comments. Many of the solutions were rather similar in nature as most verifications were the result of combining existing inequalities. In fact, Martin Lukarevski submitted two solutions of which his second is presented above.
4076. Proposed by Mehtaab Sawhney.

Prove that \((x^2 + y^2 + z^2)^3 \geq (x^3 + y^3 + z^3 + 3(\sqrt{3} - 1)xyz)^2\) for all nonnegative reals \(x, y,\) and \(z.\)

We received seven submissions, six of which were correct. We present the solution by Michel Bataille, expanded slightly by the editor.

Note first that equality holds if \(x = y = z = 0.\) Now suppose \(x + y + z > 0.\) Then by homogeneity we may assume that \(x + y + z = 1.\) Let \(m = xy + yz + zx\) and \(k = xyz.\) Then \(x^2 + y^2 + z^2 = 1 - 2m\) and

\[x^3 + y^3 + z^3 = 3xyz + (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = 1 - 3m + 3k.\]

The given inequality is equivalent, in succession, to

\[\begin{align*}
(1 - 2m)^3 &\geq (1 - 3m + 3\sqrt{3}k)^2 \\
1 - 6m + 12m^2 - 8m^3 &\geq 1 + 9m^2 + 27k^2 - 6m + 6\sqrt{3}k - 18\sqrt{3}mk \\
3m^2 + 18\sqrt{3}mk - 8m^3 - 27k^2 - 6\sqrt{3}k &\geq 0.
\end{align*}\]  

(1)

Note that \(1 - 3m \geq 0\) since \(x^2 + y^2 + z^2 \geq xy + yz + zx.\) We set \(u = \sqrt{1 - 3m}\) so \(m = \frac{1}{3}(1 - u^2)\) and then (1) becomes

\[\begin{align*}
\frac{8}{27}(1 - u^3)^3 + 27k^2 + 6\sqrt{3}k &\leq \frac{1}{3}(1 - u^2)^2 + 6\sqrt{3}(1 - u^2)k \\
\frac{8}{27}(1 - 3u^2 + 3u^4 - u^6) + 27k^2 + 6\sqrt{3}k &\leq \frac{1}{3}(1 - 2u^2 + u^4) + 6\sqrt{3}k - 6\sqrt{3}ku^2 \\
27 \cdot 6\sqrt{3}ku^2 + (27k)^2 &\leq 1 + 6u^2 - 15u^4 + 8u^6.
\end{align*}\]  

(2)

We now apply the following known result:

\[27k \leq (1 - u)^2(1 + 2u) = 1 - 3u^2 + 2u^3.\]  

(3)

(See the article “On a class of three-variable Inequalities” by Vo Quoc Ba Can, Mathematical Reflections, 2007, issue 2, and the proof by Paolo Perfetti in his solution to Crux Problem 3663, 38 (7), pp. 291-292.)

Using (3), we see that in order to establish (2), it suffices to prove the inequality:

\[6\sqrt{3}u^2(1 - u)^2(1 + 2u) + (1 - u)^4(1 + 2u)^2 \leq 1 + 6u^2 - 15u^4 + 8u^6.\]  

(4)

After straightforward computations, we see that (4) is successively equivalent to:

\[\begin{align*}
(1 - u)^2(6\sqrt{3}u^2(1 + 2u) + (1 - u)^2(1 + 2u)^2) &\leq (1 - u)^2(1 + 2u + 9u^2 + 16u^3 + 8u^4) \\
1 + 2u + 9u^2 + 16u^3 + 8u^4 - (6\sqrt{3}u^2 + 12\sqrt{3}u^3 + 1 + 2u - 3u^2 - 4u^3 + 4u^4) &\geq 0 \\
((12 - 6\sqrt{3})u^2 + (20 - 12\sqrt{3})u^3 + 4u^4) &\geq 0 \\
2u^2 + (10 - 6\sqrt{3})u + 6 - 3\sqrt{3} &\geq 0,
\end{align*}\]

which is true since \((10 - 6\sqrt{3})^2 - 8(6 - 3\sqrt{3}) = 32(5 - 3\sqrt{3}) < 0.\) Hence (4) is true and our proof is complete.
Let \(ABC\) be a triangle. Prove that
\[
\sin \frac{A}{2} \cdot \sin B \cdot \sin C + \sin A \cdot \sin \frac{B}{2} \cdot \sin C + \sin A \cdot \sin B \cdot \sin \frac{C}{2} \leq \frac{9}{8}.
\]

We received 15 submissions, of which 14 were correct and complete. We present the solution by Phil McCartney.

Let \(a = \pi - A\), \(b = \pi - B\), \(c = \pi - C\). Then \(a + b + c = \pi\) and \(a, b\) and \(c\) are in \((0, \frac{\pi}{2})\).

We have
\[
\sum_{\text{cyc}} \sin \frac{A}{2} \sin B \sin C = \sum_{\text{cyc}} \cos a \sin(\pi - 2b) \sin(\pi - 2c)
= \sum_{\text{cyc}} \cos a \sin(2b) \sin(2c)
= 4 \cos a \cos b \cos c \cdot \sum_{\text{cyc}} \sin a \sin b
\]

Hence it suffices to show that
\[
\cos a \cos b \cos c \leq \frac{1}{8} \quad \text{and} \quad \sum_{\text{cyc}} \sin a \sin b \leq \frac{9}{4}.
\]

Since \(\cos(t)\) is a concave function on \((0, \frac{\pi}{2})\), the AM-GM inequality followed by Jensen’s inequality yields:
\[
\cos a \cos b \cos c \leq \left(\frac{\cos a + \cos b + \cos c}{3}\right)^3 \leq \cos^3 \left(\frac{a + b + c}{3}\right) = \frac{1}{8},
\]
proving the first of the two inequalities in (†).

By Cauchy’s inequality, \(\sum \sin a \sin b \leq \sum \sin^2 a\); so, in order to conclude the second inequality also holds, it suffices to prove that \(\sum \sin^2 a \leq \frac{9}{4}\). However, one can show that \(\sum \sin^2 a = 2(1 + \cos a \cos b \cos c)\) using the fact that \(a + b + c = \pi\), and hence \(\sin(a) = \sin(b + c)\), we have
\[
\sin^2 a = (\sin b \cos c + \cos b \sin c)^2
= \sin^2 b \cos^2 c + 2 \sin b \cos b \cos c \sin c + \cos^2 b \sin^2 c
= (1 - \cos^2 b) \cos^2 c + 2 \sin b \cos b \cos c \sin c + \cos^2 b(1 - \cos^2 c)
= \cos^2 b + \cos^2 c - 2 \cos^2 b \cos^2 c + 2 \cos b \cos c \sin b \sin c
= \cos^2 b + \cos^2 c - 2 \cos b \cos c \cos(b + c)
= 1 - \sin^2 b + 1 - \sin^2 c + 2 \cos b \cos c \cos a,
\]
which can be rearranged to \(\sin^2 a + \sin^2 b + \sin^2 c = 2 + 2 \cos a \cos b \cos c\) as claimed.
Finally, using the first inequality in (†), we can conclude that \( \sum_{cyc} \sin^2 a \leq 2 + \frac{1}{4} = \frac{9}{8} \), showing the second inequality in (†).

Editor’s Comments. The inequalities in (†) are known and can be found in O. Bottema et al., Geometric inequalities, Groningen, Wolters-Noordhoff, 1969.

4078. Proposed by Michel Bataille.

Given \( \theta \) such that \( \frac{\pi}{3} \leq \theta \leq \frac{5\pi}{3} \), let \( M_0 \) be a point of a circle with centre \( O \) and radius \( R \) and \( M_k \) its image under the counterclockwise rotation with centre \( O \) and angle \( k\theta \). If \( M \) is the point diametrically opposite to \( M_0 \) and \( n \) is a positive integer, show that
\[
\sum_{k=0}^{n} MM_k \geq (2n + 1) \cdot \frac{R}{2}.
\]

We received two submissions, both correct, and feature the solution by AN-anduud Problem Solving Group.

We can assume that \( R = 1, M_0 = e^0 = 1 \), and \( M = -1 \); then \( M_k = e^{ik\theta}, k = 1, 2, \ldots, n \). Let \( e^{i\theta} = z \), so that \( MM_k = |1 - e^{ik\theta}| = |1 + z^k| \leq 1 + |z|^k = 2 \).

Thus we have
\[
\sum_{k=0}^{n} MM_k = \sum_{k=0}^{n} |1 + z^k| = \frac{1}{2} \sum_{k=0}^{n} 2 \cdot |1 + z^k|
\]
\[
\geq \frac{1}{2} \sum_{k=0}^{n} |1 + z^k|^2 = \frac{1}{2} \sum_{k=0}^{n} (1 + z^k)(1 + z^k)
\]
\[
= \frac{1}{2} \sum_{k=0}^{n} (1 + z^k)(1 + z^{-k}) = \frac{1}{2} \sum_{k=0}^{n} (2 + (z^k + z^{-k}))
\]
\[
= \sum_{k=0}^{n} \left( 1 + \frac{z^k + z^{-k}}{2} \right) = \sum_{k=0}^{n} (1 + \cos k\theta)
\]
\[
= n + \frac{1}{2} + \left( 1 + \frac{1}{2} + \sum_{k=1}^{n} \cos k\theta \right)
\]
\[
= \frac{2n + 1}{2} + \left( 1 + \frac{\sin (n + \frac{1}{2}) \theta}{2 \sin \frac{\theta}{2}} \right)
\]
\[
= \frac{2n + 1}{2} + \frac{2 \sin \frac{\theta}{2} + \sin (n + \frac{1}{2}) \theta}{2 \sin \frac{\theta}{2}}
\]
\[
\left( \frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6} \Rightarrow \sin \frac{\theta}{2} \geq \frac{1}{2} \Rightarrow 2 \sin \frac{\theta}{2} \geq 1 \right)
\]
\[
\geq \frac{2n + 1}{2} + \frac{1 + \sin (n + \frac{1}{2}) \theta}{2 \sin \frac{\theta}{2}} \geq \frac{2n + 1}{2}.
\]
Let $x, y, z > 0$ and $x + y + z = 2016$. Prove that :

$$x\sqrt{\frac{yz}{y + 2015z}} + y\sqrt{\frac{xz}{z + 2015x}} + z\sqrt{\frac{xy}{x + 2015y}} \leq \frac{2016}{\sqrt{3}}.$$

We received eleven solutions. We present 2 solutions.

Solution 1, by Titu Zvonaru.

We prove the general case. Let $x + y + z = t$, where $t \geq 1$. We have

$$\frac{yz}{y + (t-1)z} \leq \frac{(t-1)y + z}{t^2}. \quad (1)$$

Indeed, the inequality (1) is equivalent to

$$(t-1)y^2 + (t-1)z^2 + ((t-1)^2 + 1)yz \geq t^2yz \iff (t-1)(y - z)^2 \geq 0.$$

Using (1), the Cauchy-Schwarz Inequality and the known inequality

$$(x + y + z)^2 \geq 3(xy + yz + zx),$$

we obtain

$$x\sqrt{\frac{yz}{y + (t-1)z}} + y\sqrt{\frac{xz}{z + (t-1)x}} + z\sqrt{\frac{xy}{x + (t-1)z}}$$

$$\leq \frac{x}{t} \sqrt{(t-1)y + z} + \frac{y}{t} \sqrt{(t-1)z + x} + \frac{z}{t} \sqrt{(t-1)x + y}$$

$$= \frac{1}{t} \left( \sqrt{x} \sqrt{(t-1)xy + zx} + \sqrt{y} \sqrt{(t-1)y + xy} + \sqrt{z} \sqrt{(t-1)zx + yz} \right)$$

$$\leq \frac{1}{t} \left( (x + y + z)(t-1)xy + zx + (t-1)y + xy + (t-1)zx + yz \right)$$

$$= \frac{1}{t} \sqrt{t^2(xy + yz + zx)}$$

$$\leq \frac{1}{t} \cdot t \sqrt{\frac{(x + y + z)^2}{3}} = \frac{t}{\sqrt{3}}.$$

The equality holds if and only if $x = y = z = t/3$.

Solution 2, by Dionne Bailey, Elsie Campbell, and Charles Diminnie.

We will prove the following slight generalization of the given problem :

If $x, y, z > 0$ and $x + y + z = S + 1$, then

$$x\sqrt{\frac{yz}{y + Sz}} + y\sqrt{\frac{zx}{z + Sx}} + z\sqrt{\frac{xy}{x + Sy}} \leq \frac{S + 1}{\sqrt{3}}.$$
To begin, we invoke the general form of the Arithmetic - Geometric Mean Inequality which states that if \( a, b, \alpha, \beta > 0 \) and \( \alpha + \beta = 1 \), then

\[
a^{\alpha} \cdot b^{\beta} \leq \alpha a + \beta b,
\]

with equality if and only if \( a = b \). It follows from (2) that

\[
(y + Sz) (Sy + z) = (S + 1)^2 \left( \frac{1}{S + 1} y + \frac{S}{S + 1} z \right) \left( \frac{S}{S + 1} y + \frac{1}{S + 1} z \right)
\]

\[
\geq (S + 1)^2 \left( \frac{y^{\frac{1}{S+1}} z^{\frac{S}{S+1}}}{S^{\frac{1}{S+1}}} \right) \left( \frac{z^{\frac{1}{S+1}} y^{\frac{S}{S+1}}}{S^{\frac{1}{S+1}}} \right)
\]

\[
= (S + 1)^2 yz,
\]

and hence,

\[
x \sqrt{\frac{yz}{y + Sz}} \leq \frac{x}{S + 1} \sqrt{Sy + z}.
\]

(3)

Further, equality is attained in (3) if and only if \( y = z \).

Similar arguments show that

\[
y \sqrt{\frac{zx}{z + Sx}} \leq \frac{y}{S + 1} \sqrt{Sx + z},
\]

(4)

with equality if and only if \( z = x \), and

\[
z \sqrt{\frac{xy}{x + Sy}} \leq \frac{z}{S + 1} \sqrt{Sx + y},
\]

(5)

with equality if and only if \( x = y \).

Since \( f (t) = \sqrt{t} \) is strictly concave on \((0, \infty)\), we utilize conditions (3), (4), (5), the constraint equation \( x + y + z = S + 1 \), and Jensen’s Inequality to obtain

\[
x \sqrt{\frac{yz}{y + Sz}} + y \sqrt{\frac{zx}{z + Sx}} + z \sqrt{\frac{xy}{x + Sy}}
\]

\[
\leq \frac{x}{S + 1} \sqrt{Sy + z} + \frac{y}{S + 1} \sqrt{Sx + z} + \frac{z}{S + 1} \sqrt{Sx + y}
\]

\[
\leq \sqrt{\frac{x(Sy + z) + y(Sz + x) + z(Sx + y)}{S + 1}}
\]

\[
= \sqrt{xy + yz + zx}
\]

\[
\leq \sqrt{\frac{(x + y + z)^2}{3}}
\]

\[
= \frac{S + 1}{\sqrt{3}},
\]

with equality if and only if \( x = y = z = \frac{S + 1}{3} \).
4080. Proposed by Alina Sîntămărian and Ovidiu Furdui.

Let \( a, b \in \mathbb{R} \), with \( ab > 0 \). Calculate

\[
\int_{0}^{\infty} x^2 e^{-\left(ax - \frac{b}{x}\right)^2} \, dx.
\]

We received nine submissions of which five were correct and complete solutions. We present the solution by Michel Bataille.

We show that the value of the given integral \( I \) is

\[
I = \frac{\sqrt{\pi}(1 + 2ab)}{4a^3}.
\]

Recall that \( \int_{0}^{\infty} e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2} \). For later use, we also calculate \( J = \int_{0}^{\infty} t^2 e^{-t^2} \, dt \).

For \( X > 0 \), integrating by parts, we obtain

\[
\int_{0}^{X} t^2 e^{-t^2} \, dt = \frac{1}{2} \left( -t e^{-t^2} \right)_{0}^{X} + \int_{0}^{X} e^{-t^2} \, dt = \frac{1}{2} \left( -X e^{-X^2} + \int_{0}^{X} e^{-t^2} \, dt \right)
\]

and letting \( X \to \infty \),

\[
J = \frac{1}{2} \int_{0}^{\infty} e^{-t^2} \, dt = \frac{\sqrt{\pi}}{4}.
\]

The equation \( ax - \frac{b}{x} = y \) has a unique positive solution for \( x = \frac{1}{\sqrt{2}} (y + \varepsilon \sqrt{y^2 + 4ab}) \), where \( \varepsilon = 1 \) if \( a, b > 0 \) and \( \varepsilon = -1 \) if \( a, b < 0 \). The change of variables

\[
x = \frac{1}{2a} (y + \varepsilon \sqrt{y^2 + 4ab}), \quad dx = \frac{\varepsilon}{2a} \cdot \frac{y + \varepsilon \sqrt{y^2 + 4ab}}{\sqrt{y^2 + 4ab}} \, dy
\]

yields

\[
I = \frac{1}{8a^3} \int_{-\infty}^{\infty} \frac{(y + \varepsilon \sqrt{y^2 + 4ab})^3}{\sqrt{y^2 + 4ab}} \, e^{-y^2} \, dy.
\]

We expand the non-exponential factor in the integrand as

\[
\frac{(y + \varepsilon \sqrt{y^2 + 4ab})^3}{\sqrt{y^2 + 4ab}} = \frac{y^3}{\sqrt{y^2 + 4ab}} + 3\varepsilon y^2 + 3y \sqrt{y^2 + 4ab} + \varepsilon (y^2 + 4ab).
\]

Note the behaviour of the first and third terms on the right-hand side:

\[
|y| \sqrt{y^2 + 4ab} e^{-y^2} \sim \frac{|y|^3}{\sqrt{y^2 + 4ab}} e^{-y^2} \sim y^2 e^{-y^2} \text{ as } y \to \infty
\]

It follows that the integrals

\[
\int_{-\infty}^{\infty} \frac{y^3}{\sqrt{y^2 + 4ab}} \, e^{-y^2} \, dy \quad \text{and} \quad \int_{-\infty}^{\infty} 3y \sqrt{y^2 + 4ab} \, e^{-y^2} \, dy
\]

exist and are therefore zero, since each integrand is odd. Thus,

\[
I = \frac{1}{8a^3} \left( 6\varepsilon J + 2\varepsilon J + 8\varepsilon ab \int_{0}^{\infty} e^{-y^2} \, dy \right) = \frac{\sqrt{\pi}(1 + 2ab)}{4|a|^3}.
\]