THE OLYMPIAD CORNER
No. 346
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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d’une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n’importe quel problème. S’il vous plaît vous référer aux règles de soumission à l’endos de la couverture ou en ligne.

Pour faciliter l’examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 1 mai 2017.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d’avoir traduit les problèmes.

OC296. Soit \( \mathbb{N} = \{1, 2, 3, \ldots\} \). Déterminer toutes les fonctions \( f : \mathbb{N} \to \mathbb{N} \) qui satisfont à \( (n-1)^2 < f(n)f(f(n)) < n^2 + n \) pour tout \( n \) dans \( \mathbb{N} \).

OC297. Démontrer que
\[
\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n+1)^2} < n \cdot \left(1 - \frac{1}{\sqrt{2}}\right).
\]

OC298. Soit \( ABC \) un triangle aigu et \( H \) son orthocentre. Le cercle circonscrit au triangle \( ABH \) coupe la droite \( BC \) en \( D \). Les droites \( DH \) et \( AC \) se coupent en \( P \). Soit \( Q \) le centre du cercle circonscrit au triangle \( ADP \). Démontrer que le centre du cercle circonscrit au triangle \( ABH \) est situé sur le cercle circonscrit au triangle \( BDQ \).

OC299. Déterminer tous les entiers strictement positifs \( k \) pour lesquels
\[2^{(k-1)n+1}
\]n’est pas un diviseur de
\[
\frac{(kn)!}{n!}
\]pour tout entier strictement positif \( n \).

OC300. Les candidats d’un concours sont assis en \( n \) colonnes de manière à former une bonne configuration, c’est-à-dire une configuration dans laquelle on ne peut retrouver deux amis dans une même colonne. Dans une bonne configuration, il est impossible d’avoir tous les candidats assis en \( n-1 \) colonnes. Démontrer qu’il est toujours possible de choisir des candidats \( M_1, M_2, \ldots, M_n \) tels que \( M_i \) est assis
dans la $i^{\text{ème}}$ colonne pour tout $i = 1, 2, \ldots, n$ et $M_i$ est un ami de $M_{i+1}$ pour tout $i = 1, 2, \ldots, n-1$.

**OC296.** Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Determine all functions $f$, defined on $\mathbb{N}$ and taking values in $\mathbb{N}$, such that $(n-1)^2 < f(n)f(f(n)) < n^2 + n$ for every positive integer $n$.

**OC297.** Prove that
\[
\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n+1)^2} < n \cdot \left(1 - \frac{1}{\sqrt{2}}\right).\]

**OC298.** Triangle $ABC$ is an acute triangle and its orthocenter is $H$. The circumcircle of $\triangle ABH$ intersects line $BC$ at $D$. Lines $DH$ and $AC$ meet at $P$, and the circumcenter of $\triangle ADP$ is $Q$. Prove that the circumcenter of $\triangle ABH$ lies on the circumcircle of $\triangle BDQ$.

**OC299.** Find all positive integers $k$ such that for any positive integer $n$,
\[
2^{(k-1)n+1}
\]

does not divide
\[
\frac{(kn)!}{n!}.
\]

**OC300.** At a contest, all participants are sitting in $n$ columns and are forming a “good” configuration. (We define one configuration as “good” when we do not have 2 friends sitting in the same column). It is impossible for all the students to sit in $n-1$ columns in a “good” configuration. Prove that we can always choose contestants $M_1, M_2, \ldots, M_n$ such that $M_i$ is sitting in the $i$th column, for each $i = 1, 2, \ldots, n$ and $M_i$ is a friend of $M_{i+1}$ for each $i = 1, 2, \ldots, n-1$. 

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OLYMPIAD SOLUTIONS


OC236. Given a function \( f : \mathbb{R} \to \mathbb{R} \) with \( f(x)^2 \leq f(y) \) for all \( x, y \in \mathbb{R} \), \( x > y \), prove that \( f(x) \in [0, 1] \) for all \( x \in \mathbb{R} \).

*Originally problem 2 from day 1 of the 2014 Grade 10 All Russian Math Olympiad.*

We received 2 correct submissions. We present the solution by Michel Bataille.

Let \( x \in \mathbb{R} \). Pick some real \( a \) such that \( a > x \). Then \( f(a)^2 \leq f(x) \) so that \( f(x) \geq 0 \).

Thus \( f(x) \geq 0 \) for all \( x \in \mathbb{R} \).

Consider \( y < 0 \) and, for each integer \( n \geq 2 \), choose \( y_1, y_2, \ldots, y_{n-1} \) such that \( y > y_1 > y_2 > \cdots > y_{n-1} > 2y \) (for example take \( y_k = y + \frac{ky}{n}, \ k = 1, 2, \ldots, n-1 \)). Then we have

\[ f(y)^2 \leq f(y_1), f(y_1)^2 \leq f(y_2), \ldots, f(y_{n-1})^2 \leq f(2y), \]

from which we successively deduce:

\[ f(y)^4 \leq f(y_2), f(y)^8 \leq f(y_3), \ldots, f(y)^{2^n} \leq f(2y). \]

Thus, the sequence \( (f(y)^{2^n})_{n \geq 2} \) is bounded above (by \( f(2y) \)). It follows that \( f(y) \leq 1 \) (otherwise \( \lim_{n \to \infty} (f(y)^{2^n}) = \infty \)). Thus \( f(y) \leq 1 \) for all \( y < 0 \).

In addition, if \( x \geq 0 \), then \( x > b \) where \( b \) is a negative real number, therefore \( f(x)^2 \leq f(b) \leq 1 \) and so \( f(x) \leq 1 \) (since \( f(x) \geq 0 \)). We may conclude that \( f(x) \in [0, 1] \) for all \( x \in \mathbb{R} \).

OC237. Let \( n \) be a positive integer, and let \( S \) be the set of all integers in \( \{1, 2, \ldots, n\} \) which are relatively prime to \( n \). Set \( S_1 = S \cap (0, \frac{n}{2}], \ S_2 = S \cap (\frac{n}{2}, \frac{3n}{4}], \ S_3 = S \cap (\frac{3n}{4}, n] \). If the cardinality of \( S \) is a multiple of 3, prove that \( S_1, S_2, S_3 \) have the same cardinality.

*Originally problem 8 from day 2 of the 2014 China Girls Math Olympiad.*

No submitted solutions.

OC238. Let \( a, b, c \) be positive reals such that \( a + b + c = 3 \). Prove:

\[ \frac{a^2}{a + \sqrt{bc}} + \frac{b^2}{b + \sqrt{ca}} + \frac{c^2}{c + \sqrt{ab}} \geq \frac{3}{2} \]

and determine when equality holds.

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Originally problem 5 from day 2 of the 2014 Mexico National Olympiad.
We received 8 correct submissions. We present the solution by Ali Adnan.

From the AM-GM Inequality \( \sqrt[3]{bc} \leq \frac{b + c + 1}{3} \) and from two other such inequalities, we get

\[
\frac{a^2}{a + \sqrt[3]{bc}} + \frac{b^2}{b + \sqrt[3]{ca}} + \frac{c^2}{c + \sqrt[3]{ab}} \geq \frac{a^2}{a + \frac{b + c + 1}{3}} + \frac{b^2}{b + \frac{c + a + 1}{3}} + \frac{c^2}{c + \frac{a + b + 1}{3}}
\]

\[
\geq \frac{(a + b + c)^2}{a + b + c + \frac{2(a + b + c) + 3}{3}} = \frac{3}{2},
\]

where the last inequality is the Cauchy-Schwarz Inequality. Equality holds iff \( a = b = c = 1 \).

Editor’s Note. In this form, the Cauchy-Schwarz Inequality is also known as Titu’s Lemma or alternatively Engel’s form of the Cauchy-Schwarz Inequality.

**OC239.** In a school, there are \( n \) students and some of them are friends of each other. (Friendship is mutual.) Define \( a, b \) to be the minimum values which satisfy the following conditions:

1. We can divide students into \( a \) teams such that two students in the same team are always friends.
2. We can divide students into \( b \) teams such that two students in the same team are never friends.

Find the maximum value of \( N = a + b \) in terms of \( n \).

Originally problem 3 of the 2014 Japan Mathematical Olympiad Finals.

We present the solution by Oliver Geupel. There were no other submissions.

The problem seems to ask for the maximum value of the sum of the chromatic numbers of two complementary graphs of order \( n \). Two graphs \( A \) and \( B \) with the same set of vertices are called complementary if, for every two distinct vertices \( v \) and \( w \), exactly one of the graphs \( A \) or \( B \) has an edge \( \{v, w\} \). The chromatic number of a graph is the smallest number of colors needed to color its vertices so that no two adjacent vertices share the same color. We will show that the desired maximum value is \( n + 1 \). This is a well-known result (see E.A. Nordhaus, J.W. Gaddum, *On complementary graphs*, American Mathematical Monthly 63 (1956), 175-177). We reproduce the proof from the reference.

Denote the following assertion by \( P(n) \): If \( A \) and \( B \) are two complementary graphs of order \( n \) with chromatic numbers \( a \) and \( b \), then \( a + b \leq n + 1 \). We prove \( P(n) \) for all \( n \geq 1 \) by mathematical induction.

The base case \( P(1) \) is clearly true. Let us assume \( P(n - 1) \) for some \( n \geq 2 \). We are going to prove \( P(n) \). Withdraw one of the \( n \) vertices, say vertex \( v \), and consider

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the restrictions of graphs $A$ and $B$ to the remaining $n-1$ vertices. The sum of their chromatic numbers, say $a'$ and $b'$, is not greater than $n$. Also $a \leq a' + 1$ and $b \leq b' + 1$. As a consequence, we will have $a + b > n + 1$ only if $a' + b' = n$ and $a = a' + 1$ and $b = b' + 1$. But if $a = a' + 1$ and $b = b' + 1$ then the order of vertex $v$ in graph $A$ is at least $a'$ and the order of $v$ in $B$ is at least $b'$, so that $a' + b' \leq n - 1$. Hence $a + b \leq n + 1$, which proves $P(n)$.

The following example shows that the bound is sharp: If $A$ is the complete graph of order $n$, then $a = n$ and $b = 1$.

**OC240.** Let $ABC$ be a triangle with incenter $I$, and suppose the incircle is tangent to $CA$ and $AB$ at $E$ and $F$. Denote by $G$ and $H$ the reflections of $E$ and $F$ over $I$. Let $Q$ be the intersection of $BC$ with $GH$, and let $M$ be the midpoint of $BC$. Prove that $IQ$ and $IM$ are perpendicular.

*Originally problem 3 from day 1 of the 2014 Taiwan Team Selection Test Round 1 Mock IMO.*

*We received 8 correct submissions. We present the solution by Somasundaram Muralidharan.*

Without loss of generality, we can assume that the incenter $I$ is at the origin and the in-radius is 1. We write $\bar{z}$ to denote the complex conjugate of the complex number $z$.

Let the points of tangency $D, E, F$ of the incircle with the sides $BC, CA, AB$ respectively be represented by the complex numbers $\alpha, \beta, \gamma$. The equation of $BC$ is $z + \alpha^2 \bar{z} = 2\alpha$. Since $G$ is the reflection of $E$ in $I$, the origin, it follows that $G$ is represented by the complex number $-\beta$ and similarly $H$ is represented by $-\gamma$. 

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the complex number $-\gamma$. Thus the equation of $GH$ is $z + \beta = -\beta \gamma (\bar{z} + \bar{\beta})$. Let $q$ denote the complex number representing $Q$. Solving the equations of $BC$ and $GH$ we obtain

$$\bar{q} = \frac{2\alpha + \beta + \gamma}{\alpha^2 - \beta \gamma}$$

It is easy to see that $B$ is represented by the complex number $\frac{2\alpha \gamma}{\alpha + \gamma}$ and $C$ by the complex number $\frac{2\alpha \beta}{\alpha + \beta}$. Thus the complex number $m$ representing the mid point $M$ of $BC$ is given by

$$m = \frac{\alpha \gamma}{\alpha + \gamma} + \frac{\alpha \beta}{\alpha + \beta}$$

To show that $IM$ and $IQ$ are perpendicular, it is enough to show that $\bar{I}M$ and $\bar{I}Q$ are perpendicular where $\bar{M}$ and $\bar{Q}$ are the reflections of $M$ and $Q$ in the real axis. Note that the complex numbers representing $M$ and $Q$ are respectively $\bar{m}$ and $\bar{q}$. The conjugate $\bar{m}$ is given by

$$\bar{m} = \frac{\bar{\alpha} \bar{\gamma}}{\bar{\alpha} + \gamma} + \frac{\bar{\alpha} \bar{\beta}}{\bar{\alpha} + \beta} = \frac{1}{\bar{\alpha} + \gamma} + \frac{1}{\bar{\alpha} + \beta} = \frac{1}{\alpha + \gamma} + \frac{1}{\alpha + \beta}$$

The slope of $\bar{I}M$ is given by

$$u = \frac{\bar{m}}{m} = \frac{\frac{1}{\alpha + \gamma} + \frac{1}{\alpha + \beta}}{\frac{\alpha \gamma}{\alpha + \gamma} + \frac{\alpha \beta}{\alpha + \beta}} = \frac{2\alpha + \beta + \gamma}{\alpha(\alpha \beta + 2\beta \gamma + \gamma \alpha)}$$

Similarly the slope of $\bar{I}Q$ is given by

$$v = \frac{\bar{q}}{q} = \frac{\frac{2\alpha + \beta + \gamma}{\alpha^2 - \beta \gamma}}{\frac{2\alpha + \beta + \gamma}{\alpha^2 - \beta \gamma}} = -\frac{2\alpha + \beta + \gamma}{\alpha(\alpha \beta + 2\beta \gamma + \gamma \alpha)}$$

Since $u + v = 0$ it follows that $\bar{I}M$ and $\bar{I}Q$ are perpendicular.