OLEYMPIAD SOLUTIONS


OC236. Given a function $f: \mathbb{R} \to \mathbb{R}$ with $f(x)^2 \leq f(y)$ for all $x, y \in \mathbb{R}, x > y$, prove that $f(x) \in [0, 1]$ for all $x \in \mathbb{R}$.

Originally problem 2 from day 1 of the 2014 Grade 10 All Russian Math Olympiad.

We received 2 correct submissions. We present the solution by Michel Bataille.

Let $x \in \mathbb{R}$. Pick some real $a$ such that $a > x$. Then $f(a)^2 \leq f(x)$ so that $f(x) \geq 0$.

Thus $f(x) \geq 0$ for all $x \in \mathbb{R}$.

Consider $y < 0$ and, for each integer $n \geq 2$, choose $y_1, y_2, \ldots, y_{n-1}$ such that $y > y_1 > y_2 > \cdots > y_{n-1} > 2y$ (for example take $y_k = y + \frac{k}{n}$, $k = 1, 2, \ldots, n-1$).

Then we have

$$f(y)^2 \leq f(y_1), f(y_1)^2 \leq f(y_2), \ldots, f(y_{n-1})^2 \leq f(2y),$$

from which we successively deduce :

$$f(y)^4 \leq f(y_2), f(y)^8 \leq f(y_3), \ldots, f(y)^{2^n} \leq f(2y).$$

Thus, the sequence $(f(y)^{2^n})_{n \geq 2}$ is bounded above (by $f(2y)$). It follows that $f(y) \leq 1$ (otherwise $\lim_{n \to \infty} (f(y))^{2^n} = \infty$). Thus $f(y) \leq 1$ for all $y < 0$.

In addition, if $x \geq 0$, then $x > b$ where $b$ is a negative real number, therefore $f(x)^2 \leq f(b) \leq 1$ and so $f(x) \leq 1$ (since $f(x) \geq 0$). We may conclude that $f(x) \in [0, 1]$ for all $x \in \mathbb{R}$.

OC237. Let $n$ be a positive integer, and let $S$ be the set of all integers in $\{1, 2, \ldots, n\}$ which are relatively prime to $n$. Set $S_1 = S \cap \left[0, \frac{n}{3}\right], S_2 = S \cap \left(\frac{n}{3}, \frac{2n}{3}\right], S_3 = S \cap \left(\frac{2n}{3}, n\right]$. If the cardinality of $S$ is a multiple of 3, prove that $S_1, S_2, S_3$ have the same cardinality.

Originally problem 8 from day 2 of the 2014 China Girls Math Olympiad.

No submitted solutions.

OC238. Let $a, b, c$ be positive reals such that $a + b + c = 3$. Prove :

$$\frac{a^2}{a + \sqrt{bc}} + \frac{b^2}{b + \sqrt{ca}} + \frac{c^2}{c + \sqrt{ab}} \geq \frac{3}{2}$$

and determine when equality holds.
Originally problem 5 from day 2 of the 2014 Mexico National Olympiad.

We received 8 correct submissions. We present the solution by Ali Adnan.

From the AM-GM Inequality \( \sqrt[3]{bc} \leq \frac{b + c + 1}{3} \) and from two other such inequalities, we get

\[
\frac{a^2}{a + \sqrt[3]{bc}} + \frac{b^2}{b + \sqrt[3]{ca}} + \frac{c^2}{c + \sqrt[3]{ab}} \geq \frac{(a + b + c)^2}{a + b + c + \frac{2(a+b+c)+3}{3}} = \frac{3}{2},
\]

where the last inequality is the Cauchy-Schwarz Inequality. Equality holds iff \( a = b = c = 1 \).

Editor’s Note. In this form, the Cauchy-Schwarz Inequality is also known as Titu’s Lemma or alternatively Engel’s form of the Cauchy-Schwarz Inequality.

**OC239.** In a school, there are \( n \) students and some of them are friends of each other. (Friendship is mutual.) Define \( a, b \) to be the minimum values which satisfy the following conditions:

1. We can divide students into \( a \) teams such that two students in the same team are always friends.
2. We can divide students into \( b \) teams such that two students in the same team are never friends.

Find the maximum value of \( N = a + b \) in terms of \( n \).

Originally problem 3 of the 2014 Japan Mathematical Olympiad Finals.

We present the solution by Oliver Geupel. There were no other submissions.

The problem seems to ask for the maximum value of the sum of the chromatic numbers of two complementary graphs of order \( n \). Two graphs \( A \) and \( B \) with the same set of vertices are called complementary if, for every two distinct vertices \( v \) and \( w \), exactly one of the graphs \( A \) or \( B \) has an edge \( \{v, w\} \). The chromatic number of a graph is the smallest number of colors needed to color its vertices so that no two adjacent vertices share the same color. We will show that the desired maximum value is \( n + 1 \). This is a well-known result (see E.A. Nordhaus, J.W. Gaddum, *On complementary graphs*, American Mathematical Monthly 63 (1956), 175-177). We reproduce the proof from the reference.

Denote the following assertion by \( P(n) \) : If \( A \) and \( B \) are two complementary graphs of order \( n \) with chromatic numbers \( a \) and \( b \), then \( a + b \leq n + 1 \). We prove \( P(n) \) for all \( n \geq 1 \) by mathematical induction.

The base case \( P(1) \) is clearly true. Let us assume \( P(n-1) \) for some \( n \geq 2 \). We are going to prove \( P(n) \). Withdraw one of the \( n \) vertices, say vertex \( v \), and consider.
the restrictions of graphs $A$ and $B$ to the remaining $n - 1$ vertices. The sum of their chromatic numbers, say $a'$ and $b'$, is not greater than $n$. Also $a \leq a' + 1$ and $b \leq b' + 1$. As a consequence, we will have $a + b > n + 1$ only if $a' + b' = n$ and $a = a' + 1$ and $b = b' + 1$. But if $a = a' + 1$ and $b = b' + 1$ then the order of vertex $v$ in graph $A$ is at least $a'$ and the order of $v$ in $B$ is at least $b'$, so that $a' + b' \leq n - 1$. Hence $a + b \leq n + 1$, which proves $P(n)$.

The following example shows that the bound is sharp: If $A$ is the complete graph of order $n$, then $a = n$ and $b = 1$.

**OC240.** Let $ABC$ be a triangle with incenter $I$, and suppose the incircle is tangent to $CA$ and $AB$ at $E$ and $F$. Denote by $G$ and $H$ the reflections of $E$ and $F$ over $I$. Let $Q$ be the intersection of $BC$ with $GH$, and let $M$ be the midpoint of $BC$. Prove that $IQ$ and $IM$ are perpendicular.

*Originally problem 3 from day 1 of the 2014 Taiwan Team Selection Test Round 1 Mock IMO.*

We received 8 correct submissions. We present the solution by Somasundaram Muralidharan.

Without loss of generality, we can assume that the incenter $I$ is at the origin and the in-radius is 1. We write $\bar{z}$ to denote the complex conjugate of the complex number $z$.

Let the points of tangency $D, E, F$ of the incircle with the sides $BC, CA, AB$ respectively be represented by the complex numbers $\alpha, \beta, \gamma$. The equation of $BC$ is $\bar{z} + \alpha \bar{z} = 2\alpha$. Since $G$ is the reflection of $E$ in $I$, the origin, it follows that $G$ is represented by the complex number $-\beta$ and similarly $H$ is represented by $-\gamma$. The coordinates of $I, E, G, H, Q$ are $\frac{\alpha + \beta}{2}, \frac{\alpha}{2}, -\frac{\beta}{2}, -\frac{\gamma}{2}$ respectively.
the complex number $-\gamma$. Thus the equation of $GH$ is $z + \beta = -\beta\gamma(\bar{z} + \bar{\beta})$. Let $q$ denote the complex number representing $Q$. Solving the equations of $BC$ and $GH$ we obtain

$$\bar{q} = \frac{2\alpha + \beta + \gamma}{\alpha^2 - \beta\gamma}$$

It is easy to see that $B$ is represented by the complex number $\frac{2\alpha\gamma}{\alpha + \gamma}$ and $C$ by the complex number $\frac{2\alpha\beta}{\alpha + \beta}$. Thus the complex number $m$ representing the mid point $M$ of $BC$ is given by

$$m = \frac{\alpha\gamma}{\alpha + \gamma} + \frac{\alpha\beta}{\alpha + \beta}$$

To show that $IM$ and $IQ$ are perpendicular, it is enough to show that $\bar{I}\bar{M}$ and $\bar{I}\bar{Q}$ are perpendicular where $\bar{M}$ and $\bar{Q}$ are the reflections of $M$ and $Q$ in the real axis. Note that the complex numbers representing $M$ and $Q$ are respectively $\bar{m}$ and $\bar{q}$. The conjugate $\bar{m}$ is given by

$$\bar{m} = \frac{\bar{\alpha}\gamma}{\bar{\alpha} + \bar{\gamma}} + \frac{\bar{\alpha}\bar{\beta}}{\bar{\alpha} + \bar{\beta}} = \frac{1}{\frac{\alpha}{\alpha + \gamma}} + \frac{1}{\frac{\beta}{\alpha + \beta}} = \frac{1}{\frac{\alpha}{\alpha + \gamma}} + \frac{1}{\frac{\alpha}{\alpha + \beta}}$$

The slope of $I\bar{M}$ is given by

$$u = \frac{\bar{m}}{m} = \frac{\frac{1}{\alpha + \gamma} + \frac{1}{\alpha + \beta}}{\frac{\alpha}{\alpha + \gamma} + \frac{\beta}{\alpha + \beta}} = \frac{2\alpha + \beta + \gamma}{\alpha(\alpha\beta + 2\beta\gamma + \gamma\alpha)}$$

Similarly the slope of $I\bar{Q}$ is given by

$$v = \frac{\bar{q}}{q} = \frac{\frac{2\alpha + \beta + \gamma}{\alpha^2 - \beta\gamma}}{\frac{2\alpha + \beta + \gamma}{\alpha^2 - \beta\gamma}} = -\frac{2\alpha + \beta + \gamma}{\alpha(\alpha\beta + 2\beta\gamma + \gamma\alpha)}$$

Since $u + v = 0$ it follows that $I\bar{M}$ and $I\bar{Q}$ are perpendicular.