Where did the complex plane come from?
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Around 800 AD (184 AH) the first text on algebra was written by a man whose name tells us that he came from the area south of the Sea of Azov, currently Uzbekistan. Al-Khwarizmi’s name eventually turned into the word “algorithm” and he appears to have invented the term algebra to describe the art of finding an unknown quantity by a key trick: you assume that you know what it is – we call it $x$ or $y$, though he used a word that translates as thing – and then, so to speak, work backwards from what you do know about the unknown thing to determine, if possible, what its value is.

Al-Khwarizmi’s text – which arrived in Europe about 400 years later – shows how to solve linear and quadratic equations. There were many differences between how he did it and how we do it now, but one important one was that in the old days there were no negative numbers allowed. As other people came to work with these techniques over the years, the number system gradually became extended to allow negative results. For a long time these were treated as a bit suspect, so that the solutions to $x^2 + x - 6 = 0$ would be described as a “true” root, $x = 2$, and a “false” root, $x = -3$.

By the 1500s in Europe many experts had mastered these techniques, described in printed books and extended, for example including the solution of cubic equations. Cardano’s *Ars magna* (Great Art), published in 1545, described how to solve quadratics, cubics, and quartics (degree 4 polynomial equations). In doing so, it emphasized that negative numbers were not the only inconvenient quantity that arises when algebra is done. We know already from the quadratic formula that the square roots of negative numbers come up routinely. Such numbers were worse than “false”: they were often called “impossible”.

Negative numbers are relatively easy to reconcile with common sense. Often, for example, they are treated in monetary terms, as a debt (and written in red ink in accounting ledgers). They remain hard to interpret geometrically, though: what is meant by a negative area? Conventionally, it can be one you subtract from something bigger, and mathematical writers became used to such ideas, which go along with our ideas of the number line with a 0 in the middle (whatever the middle of an infinite line means).

The square roots of negative numbers don’t easily make sense, though. Whether arithmetically or geometrically or as ratios they are hard to interpret until we get them into the right context. But even in the 1500s, Cardano and others saw that such quantities can indeed be used to calculate results that do make sense.

For example, Cardano’s cubic formula for the case $x^3 = cx + d$ has the form

$$x = \sqrt[3]{d/2 + \sqrt{(d/2)^2 - (c/3)^3}} + \sqrt[3]{d/2 - \sqrt{(d/2)^2 - (c/3)^3}}.$$
We won’t derive the formula here, but as an exercise you can check that it will work. Note in passing that cubics always have at least one real root.

If you apply this to the equation \( x^3 = 15x + 4 \), you get an “impossible” number, one we would call complex:

\[
x = \sqrt{2 + \sqrt{-121}} + \sqrt{2 - \sqrt{-121}}.
\]

Now, it is easy to see that \( x = 4 \) satisfies the given equation. Could it be that the complex number we found by the formula is really just a complicated way to write the number 4? A mathematician called Bombelli thought so, and gave an argument to support his idea. Writing \( a + \sqrt{-b} = \sqrt{2 + \sqrt{-121}} \), he then has \( a - \sqrt{-b} = \sqrt{2 - \sqrt{-121}} \), so that \( x = 2a \). But it’s easy to show that \( a^2 + b = 5 \), \( a^3 - 3ab = 2 \), and so for \( a \) we need a number with square less than 5 and cube greater than 2. Now, \( a = 2 \) is such a number, and the value \( b = 1 \) satisfies the equations. Putting all this together, \( x = 4 \). This made it seem that there was some value of thinking of these numbers in the form \( a + \sqrt{b} \).

Because of this idea that one could do good things mathematically with imaginary numbers, researchers kept trying to come up with interpretations that would make some kind of sense to the average educated person. Being able to see, or picture, what is going on in a mathematical situation often allows you to determine in which direction to move to proceed methodically towards a rigorous solution.

Into our picture of how to interpret imaginary numbers comes a French bookstore owner named Jean-Robert Argand (1768-1822). To see how powerful a good interpretation can be, consider the following:

Take a positive integer, for example 2. Now multiply this number by \(-1\). The result, of course, is \(-2\). But let’s think carefully about this. The magnitude of the number hasn’t changed – it is still 2 units from the origin but upon drawing a line from the origin to the number, we observe that we are pointing in the opposite direction. So, we could say that multiplying by \(-1\) flipped the number to the opposite direction or we could say that multiplying by \(-1\) rotated the line joining the origin to 2 by 180 degrees. This second idea is powerful.

Working in the early 1800s, Argand wanted to graph complex numbers, and he did so in a two-dimensional plane with the real part of the number on the \(x\)-axis and the imaginary part on the \(y\)-axis (this is now called the Argand plane or the complex plane) and he interpreted multiplying by \(i\) as a rotation of ninety degrees, where the imaginary unit \(i\) is the square root of \(-1\). Multiplying by \(-1\) rotated by 180 degrees, so this was consistent with the algebraic situation. [Ed. Warning: the term “complex plane” is common, but one mathematician’s plane is another mathematician’s line.]

Without further knowledge, ask yourself: what is the square root of \(i\)? This seems very difficult but, using the geometrical interpretation of Argand, we should conclude that can be found from \(+1\) by a rotation of 45 degrees. Take the unit vector \((1, 0)\) and rotate it 45 degrees – where are you in terms of the \(x\) and \(y\)
coordinates? Think trigonometry and the unit circle. You are at \( \sin(45^\circ) \) on the
\( y \)-axis and \( \cos(45^\circ) \) on the \( x \)-axis. This would mean that:

\[
\sqrt{i} = \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}.
\]

No algebra at all! Just Argand’s geometrical insight.

But we can use algebra to check this. Before you read on, try it yourself, remembering to use the idea that \( i^2 = -1 \) to simplify when possible.

Tried it? We have

\[
\left( \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2} + 2i \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{i^2}{2} = i.
\]

In fact, Argand was not the first person to come up with the idea of representing a complex number as a point in a two-dimensional plane. He was preceded by a Norwegian land surveyor, Caspar Wessel. But Mr. Wessel wrote in Danish in a journal not much read and, though his work was published 9 years before Argand’s, it went unnoticed.

This geometrical interpretation of complex numbers was independently discovered for a third time by a titan of mathematics – a figure who towers above both Wessel and Argand in mathematical creativity. In 1831, Carl Friedrich Gauss (1777-1855) came up with the idea again. In Germany, in fact, the complex plane, where the point \( (a,b) \) is associated with the number \( a + ib \), is called the Gaussian plane.

Now the idea of taking roots in the complex plane is tied to whether or not you can construct a regular polygon with a given number of sides using simply a straight edge and compass. Try, for example, to inscribe an equilateral triangle or a hexagon inside the unit circle using only a compass and ruler (but the ruler can only be used to draw straight lines – it cannot be used to measure anything).
It turns out that you cannot always do this – it will only work for certain regular polygons. It is not easy to figure out which ones, but that is exactly what Gauss did at age 19(!) (The proof was actually completed later by Wantzel.) Gauss and Wantzel showed it is possible to construct the regular polygon using only compass and straightedge exactly when the number of sides is (a) a power of 2 (greater than 1) or (b) the product of a power of 2 (possibly 0) and distinct Fermat primes. These are prime numbers of the form $2^{2^n} + 1$. Of numbers having this form, these 5 are known to be prime: 3, 5, 17, 257, 65537.

While it was then known that it is possible to do so, it doesn’t mean that anyone knew how to use the compass and straightedge to do so for any but the smallest number of sides. Gauss, in fact, figured out how to do this for the 17-gon. He was so proud of this that he requested that the 17-sided regular polygon be inscribed on his tombstone. The stone mason declined. Too hard to do? Or would it just look like a circle given the precision of working in stone? Both the others have since been constructed: in 1894 J. G. Hermes completed the 65537-gon construction in a 200-page manuscript.

Let’s conclude by looking at the relationship between the regular pentagon and a plot in the complex plane. This turns out to be related to solving the (complex) equation $z^5 = 1$. Here we will use the polar form of the complex number:

$$z = x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

where $r$ is the modulus of the complex number (the length of the vector joining the point to the origin) and $\theta$ is the angle between the positive real axis (the $x$-axis) and the same vector. It is easy to see that $r^2 = x^2 + y^2$, and you can see also that $x = r \cos(\theta)$ and $y = r \sin(\theta)$. The relationship between the trigonometric functions and the exponential function can be taken as a definition, though there is a good reason for it.
Then we have
\[ z^5 = r^5 e^{5i\theta} = r^5 (\cos(5\theta) + i \sin(5\theta)) = 1 + 0i \]
meaning that \( r = 1, \cos(5\theta) = 1, \) and \( \sin(5\theta) = 0, \) since we equate real and imaginary parts to solve. This means that \( 5\theta = 0, 2\pi, 4\pi, \ldots \) so that we get a complete set of points on the graph if we have \( \theta = 2k\pi/5, \) where \( k = 0, 1, 2, 3, 4. \) If you plot these points you see that we have the vertices of a regular pentagon consisting of points on the unit circle at intervals of 72°. Gauss’ method was called \textit{cyclotomy}, which means “circle-cutting”, so this shows us where that term comes from and what it has to do with the Gaussian plane.

**Exercise 1**
Use geometric reasoning similar to how we found \( \sqrt{i} \) to find a 4\textsuperscript{th} root of \( i. \)

**Exercise 2**
Consider the 2-dimensional domain shown in the graph below:

Find the image of this domain under the function \( f(z) = 2iz. \) (Suggestion: Draw another copy of the complex plane to plot the image).

**Exercise 3**
Using the fact that \( \sqrt{i} = \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}, \) find the coefficients of the 2 × 2 matrix that will rotate any given vector by 45 degrees.