SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015 : 41(7), p. 302–305.

4061. *Proposed by Leonard Giugiuc.*

Let ABC be a non-obtuse triangle none of whose angles are less than $\frac{\pi}{4}$. Find the minimum value of $\sin A \sin B \sin C$.

We received twelve submissions, of which 10 were correct and one was faulty. We present two solutions.

Solution 1, by Adnan Ali.

Let us first fix angle A and determine the values of B and C for which the product $\sin A \sin B \sin C$ is smallest. Since A is fixed, this product is minimized if and only if

$$\sin B \sin C = \frac{\cos(B-C) - \cos(B+C)}{2} = \frac{\cos(B-C) - \cos(\pi - A)}{2}$$

is minimized. But $\cos(\pi - A)$ is fixed and so it is enough to minimize $\cos(B-C)$. Because the triangle is not obtuse and all angles are not less than $\pi/4$, we have $|B - C| \le \pi/4$. Since the cosine function is decreasing over $[0, \pi/2]$, $\cos(B - C)$ is minimized if $|B - C|$ is maximum, and that happens when $B = \pi/4$ and $C = 3\pi/4 - A$, or vice-versa. So now we have reduced the problem of finding the minimum value of the given product to finding the minimum value of

$$\sin A \sin(\pi/4) \sin(3\pi/4 - A), \quad \pi/4 \le A \le \pi/2.$$

This is quickly done by minimizing $\sin A \sin(3\pi/4 - A) = \frac{\cos(3\pi/4-2A)-\cos(3\pi/4)}{2}$, which is same as minimizing $\cos(3\pi/4-2A)$, where $\pi/4 \le A \le \pi/2$. The bounds on A imply that $-\pi/4 \le 3\pi/4 - 2A \le \pi/4$, and so the minimum value of $\cos(3\pi/4 - 2A)$ is achieved for $3\pi/4 - 2A = -\pi/4$ or $\pi/4$; each of these values leads to an isosceles right triangle. Thus, the minimum value of $\sin A \sin B \sin C$ is $1/2$, achieved for an isosceles right triangle ABC.

Solution 2, by Daniel Dan.

We use the identity $\sin A \sin B \sin C = \frac{1}{4}(\sin 2A + \sin 2B + \sin 2C)$. Define

$$f(x) : \left[\frac{\pi}{2}, \pi\right] \to [0,1], \quad f(x) = \sin x,$$

and note that the function is concave; in particular, every point of the graph of $f(x)$ except for its end points, namely $\left(\frac{\pi}{2}, 1\right)$ and $(\pi, 0)$, lies above the line

$$g(x) = -\frac{2}{\pi}x + 2$$

that joins those end points. Consequently, we have

$$\begin{aligned}\frac{1}{4}\left((f(2A)+f(2B)+f(2C)\right) &\geq \frac{1}{4}\left((g(2A)+g(2B)+g(2C)\right)\\ &= \frac{1}{4}\left(-\frac{2(2A+2B+2C)}{\pi}+6\right)=\frac{1}{2}.\end{aligned}$$

We conclude that the product $\sin A \sin B \sin C$ cannot be less than $\frac{1}{2}$ when all three angles are restricted to the domain $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$. The minimum is achieved if and only if $f(x)=g(x)$ for x equal to $2A, 2B$, and $2C$; because $A+B+C=\pi$, this is possible only if one of the angles is $\frac{\pi}{2}$ while the other two are $\frac{\pi}{4}$.

4062. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let L_n denote the nth Lucas number defined by $L_0=2, L_1=1$ and $L_{n+2}=L_{n+1}+L_n$ for all $n\geq 0$. Prove that

$$\frac{L_n^4+L_{n+1}^4}{L_nL_{n+1}}+\frac{L_{n+1}^4+L_{n+3}^4}{L_{n+1}L_{n+3}}+\frac{L_{n+3}^4+L_n^4}{L_{n+3}L_n}\geq\frac{2}{3}L_{n+4}^2.$$

We received ten correct and complete solutions. We present the solutions of Arkady Alt, who like most submitters used standard inequalities for a simple proof, and a slightly modified version of the solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, who made heavier use of the given recursion to find a stronger bound.

Solution 1, by Arkady Alt.

Since $a^4+b^4\geq ab(a^2+b^2)$ (as this can be rewritten as $(a^2+ab+b^2)(a-b)^2\geq 0$) and $a^2+b^2+c^2\geq\frac{(a+b+c)^2}{3}$ for all $a,b,c\in\mathbb{R}$, we have

$$\begin{aligned}&\frac{L_n^4+L_{n+1}^4}{L_nL_{n+1}}+\frac{L_{n+1}^4+L_{n+3}^4}{L_{n+1}L_{n+3}}+\frac{L_{n+3}^4+L_n^4}{L_{n+3}L_n}\\ &\geq\frac{L_nL_{n+1}(L_n^2+L_{n+1}^2)}{L_nL_{n+1}}+\frac{L_{n+1}L_{n+3}(L_{n+1}^2+L_{n+3}^2)}{L_{n+1}L_{n+3}}+\frac{L_{n+3}L_n(L_{n+3}^2+L_n^2)}{L_{n+3}L_n}\\ &=2(L_n^2+L_{n+1}^2+L_{n+3}^2)\\ &\geq 2\frac{(L_n+L_{n+1}+L_{n+3})^2}{3}=\frac{2(L_{n+2}+L_{n+3})^2}{3}\\ &=\frac{2L_{n+4}^2}{3}\end{aligned}$$

Solution 2, by Dionne Bailey, Elsie Campbell, and Charles Diminnie.

More generally, we will show that for all $n\geq 0$,

$$\frac{L_n^4+L_{n+1}^4}{L_nL_{n+1}}+\frac{L_{n+1}^4+L_{n+3}^4}{L_{n+1}L_{n+3}}+\frac{L_{n+3}^4+L_n^4}{L_{n+3}L_n}>2L_{n+4}^2.$$

We can check by hand that this holds for $n \leq 2$.

For $n \geq 3$ we first use the Arithmetic Mean - Geometric Mean inequality to obtain

$$\begin{aligned} x^4 + y^4 &= 2x^2y^2 + (x^2 - y^2)^2 \\ &= 2x^2y^2 + (x+y)^2(x-y)^2 \\ &\geq 2x^2y^2 + 4xy(x-y)^2 \\ &= xy(2xy + 4(x-y)^2) \end{aligned}$$

and hence

$$\frac{x^4 + y^4}{xy} \geq 2xy + 4(x-y)^2.$$

Using this property and the recursion for the Lucas numbers (multiple times, when necessary), we get

$$\frac{L_n^4 + L_{n+1}^4}{L_nL_{n+1}} \geq 2L_nL_{n+1} + 4(L_{n+1} - L_n)^2 = 4L_{n+1}^2 - 6L_{n+1}L_n + 4L_n^2,$$

$$\frac{L_{n+1}^4 + L_{n+3}^4}{L_{n+1}L_{n+3}} \geq 2L_{n+1}L_{n+3} + 4(L_{n+3} - L_{n+1})^2 = 8L_{n+1}^2 + 10L_{n+1}L_n + 4L_n^2,$$

$$\frac{L_{n+3}^4 + L_n^4}{L_{n+3}L_n} \geq 2L_{n+3}L_n + 4(L_{n+3} - L_n)^2 = 16L_{n+1}^2 + 4L_{n+1}L_n + 2L_n^2,$$

and

$$2L_{n+4}^2 = 18L_{n+1}^2 + 24L_{n+1}L_n + 8L_n^2.$$

Combining these, we obtain

$$\begin{aligned} \frac{L_n^4 + L_{n+1}^4}{L_nL_{n+1}} + \frac{L_{n+1}^4 + L_{n+3}^4}{L_{n+1}L_{n+3}} + \frac{L_{n+3}^4 + L_n^4}{L_{n+3}L_n} &\geq 28L_{n+1}^2 + 8L_{n+1}L_n + 10L_n^2 \\ &= 2L_{n+4}^2 + 10L_{n+1}^2 - 16L_{n+1}L_n + 2L_n^2 \\ &= 2L_{n+4}^2 - 6L_{n+1}L_n + 10L_{n+1}L_{n-1} + 2L_n^2 \\ &= 2L_{n+4}^2 + 4L_{n+1}L_{n-1} - 6L_{n+1}L_{n-2} + 2L_n^2 \\ &= 2L_{n+4}^2 + 4L_{n+1}L_{n-1} - 4L_{n+1}L_{n-2} + 2L_nL_{n-1} - 2L_{n-1}L_{n-2} \\ &= 2L_{n+4}^2 + 4L_{n+1}L_{n-3} + L_{n-1}^2 \\ &> 2L_{n+4}^2. \end{aligned}$$

4063. *Proposed by Marcel Chiriţă.*

Let a, b, c be real numbers greater than or equal to 3. Show that

$$\min\left(\frac{a^2b^2 + 3b^2}{b^2 + 27}, \frac{b^2c^2 + 3c^2}{c^2 + 27}, \frac{a^2c^2 + 3a^2}{a^2 + 27}\right) \leq \frac{abc}{9}.$$

We received six submissions all of which were correct. We present a composite of the similar solutions by Arkady Alt and Leonard Guigiuc.

Suppose to the contrary that

$$\min\left(\frac{a^2b^2+3b^2}{b^2+27},\ \frac{b^2c^2+3c^2}{c^2+27},\ \frac{a^2c^2+3a^2}{a^2+27}\right) > \frac{abc}{9}.$$

Then we have $\prod_{cyc}\dfrac{a^2b^2+3b^2}{b^2+27} > \dfrac{a^3b^3c^3}{9^3}$, so $\prod_{cyc}\dfrac{a^2+3}{a^2+27} > \dfrac{abc}{9^3}$.

But since $\frac{a}{9}-\frac{a^2+3}{a^2+27}=\frac{a^3-9a^2+27a-27}{9(a^2+27)}=\frac{(a-3)^2}{9(a^2+27)}\geq 0$, we have $\frac{a^2+3}{a^2+27}\leq\frac{a}{9}$.

Similarly, $\frac{b^2+3}{b^2+27}\leq\frac{b}{9}$ and $\frac{c^2+3}{c^2+27}\leq\frac{c}{9}$.

Hence, $\prod_{cyc}\dfrac{a^2+3}{a^2+27} > \dfrac{abc}{9^3}$ is a contradiction.

4064. *Proposed by Michel Bataille.*

In the plane of a triangle ABC, let Γ be a circle whose centre O is not on the sidelines AB, BC, CA. Let A', B', C' be the poles of the lines BC, CA, AB with respect to Γ, respectively. Prove that

$$\frac{OA'\cdot B'C'}{OA\cdot BC}=\frac{OB'\cdot C'A'}{OB\cdot CA}=\frac{OC'\cdot A'B'}{OC\cdot AB}.$$

We received five solutions, all correct, and present the solution by Joel Schlosberg, slightly modified by the editor.

One way to define the pole A' of the line BC with respect to the circle Γ is by reciprocation, namely A' is the inverse in Γ of the foot of the perpendicular from O to BC. [See, for example H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited* (The Mathematical Association of America, 1967), Section 6.1.] Conversely, if D is the inverse of A in Γ, then the polar of A, namely $B'C'$, is the line through D that is perpendicular to OA. We shall use three immediate consequences of this definition. If r is the radius of Γ, then $OA\cdot OD=r^2$, or

$$OA=\frac{r^2}{OD}. \tag{1}$$

Since $OB'\perp CA$ and $OC'\perp AB$, $\angle B'OC'$ is equal to or supplementary to $\angle BAC$. Let R be the circumradius of $\triangle ABC$. By the law of sines,

$$BC=2R\sin\angle BAC=2R\sin\angle B'OC'. \tag{2}$$

Finally, since each is the area of $\triangle OB'C'$,

$$\frac{1}{2}B'C'\cdot OD=\frac{1}{2}OB'\cdot OC'\sin\angle B'OC'. \tag{3}$$

Using in turn (1) and (2), then (3), we get

$$\frac{OA'\cdot B'C'}{OA\cdot BC}=\frac{OA'\cdot B'C'}{(r^2/OD)\cdot 2R\sin\angle B'OC'}=\frac{OA'}{2Rr^2}\cdot\frac{B'C'\cdot OD}{\sin\angle B'OC'}=\frac{OA'\cdot OB'\cdot OC'}{2Rr^2}.$$

The same reasoning shows that $\dfrac{OB'\cdot C'A'}{OB\cdot CA}$ and $\dfrac{OC'\cdot A'B'}{OC\cdot AB}$ are also equal to $\dfrac{OA'\cdot OB'\cdot OC'}{2Rr^2}$.

4065. *Proposed by Martin Lukarevski.*

Let ABC be a triangle with a, b, c as lengths of its sides and let R, r, s denote the circumradius, inradius and semiperimeter, respectively. Prove that

$$\frac{1}{(s-a)^2}+\frac{1}{(s-b)^2}+\frac{1}{(s-c)^2}\geq\frac{2}{r}\left(\frac{1}{r}-\frac{1}{R}\right).$$

We received ten correct and complete solutions. We present the solution by the proposer.

We use the Garfunkel-Bankoff inequality (Problem 825, proposed by J. Garfunkel, solution by L. Bankoff, ***Crux*** 9 (1983), p.79 and 10 (1984), p.168) :

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \tag{1}$$

which by the well-known identity

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R} \tag{2}$$

is equivalent to

$$\tan^2\frac{A}{2}+\tan^2\frac{A}{2}+\tan^2\frac{A}{2}\geq 2-\frac{2r}{R}.$$

By another well-known identity, which states that

$$\frac{1}{s-a}=\frac{1}{r}\tan\frac{A}{2}, \tag{3}$$

we have that

$$\begin{aligned}\frac{1}{(s-a)^2}+\frac{1}{(s-b)^2}+\frac{1}{(s-c)^2}&=\frac{1}{r^2}\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right)\\&\geq\frac{1}{r^2}\left(2-\frac{2r}{R}\right)\\&=\frac{2}{r}\left(\frac{1}{r}-\frac{1}{R}\right),\end{aligned}$$

with equality, as in (1), only for the equilateral triangle.

4066. *Proposed by Mihaela Berindeanu.*

Prove that for $a, b, c > 0$ and $ab + ac + bc = 2016$,

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq \frac{2019^2}{2016}.$$

We received 19 solutions all of which are correct. We present a composite of nearly identical solutions by Andrea Fanchini and Titu Zvonaru.

We prove the more general result that if $a, b, c > 0$ such that $ab+bc+ca = k$, then

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq \frac{(k+3)^2}{k}.$$

Note first that the trivial inequality $x^2 + y^2 + z^2 \geq xy + yz + zx$ implies

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}. \tag{1}$$

Using (1) together with AM-GM and AM-HM inequalities we then have

$$(a+\frac{1}{b})^2+(b+\frac{1}{c})^2+(c+\frac{1}{a})^2 = a^2+b^2+c^2+2(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

$$\begin{aligned} &\geq ab+bc+ca+2\cdot 3+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca} \\ &\geq k+6+\frac{9}{ab+bc+ca} \\ &= \frac{(k+3)^2}{k} \end{aligned}$$

Editor's comments. Most of the other solutions used AM-GM, AM-QM and/or Cauchy Schwarz Inequalities. It is trivial to see that equality holds if and only if $a = b = c = \frac{\sqrt{3k}}{3}$.

4067. *Proposed by Mehtaab Sawhney.*

Consider a graph G such that between any three vertices in G there are either 0 or 2 edges. Classify all such graphs G.

We received seven correct and complete solutions. We present the solution by Joel Schlosberg.

We claim that a graph G satisfies the condition if and only if it is either an edgeless graph or a complete bipartite graph.

If G is edgeless, any three vertices have zero edges between them, so G trivially satisfies the condition. If G is a complete bipartite graph, the vertices of G can be partitioned into two sets S_1, S_2, such that two vertices are adjacent if and only if they are in different sets. If three vertices are in the same set, they have zero edges between them ; otherwise, two of them are in one set and one is in the other, leading to two edges between them. Thus G satisfies the condition.

Conversely, suppose that G satisfies the condition. If G is not edgeless, then there exist two vertices v_1, v_2 with an edge between them. For $k = 1, 2$, let S_k be the set of vertices that share an edge with v_k. Clearly v_1 is in S_2 but not S_1 and v_2 in S_1 but not S_2. If v is a vertex of G different from v_1 and v_2, then the three vertices v, v_1, v_2 must have exactly two edges between them, since they cannot have zero. Thus v is in exactly one of S_1 or S_2. Therefore S_1 and S_2 form a partition of the vertices of G. Suppose v, w are vertices in the same set, say S_1. Then there is no edge between v and w, as otherwise we would have three edges between v, w and v_1. Now suppose $v \in S_1$ and $w \in S_2$. If $v = v_2$ then there is an edge between v and w by the definition of S_2. Otherwise consider the three vertices v, w and v_2. There is an edge between w and v_2 by the definition of S_2 and no edge between v and v_2, as just shown. Therefore there must be an edge between v and w. Thus we have proven that G is a complete bipartite graph.

4068. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers. Prove that

$$\frac{a+2b}{2a+3b+c} + \frac{b+2c}{a+2b+3c} + \frac{c+2a}{3a+b+2c} \leq \frac{3}{2}.$$

Editor's comments. We received 25 submissions all of which are correct. However, it was pointed out by Michael Bataille, and Dionne Bailey, Elsie Campbell, and Charles Diminnie that this problem is the same as Crux problem #4016 (by the same proposer) which appeared on p. 74 of ***Crux*** 41 (2). The only difference being that in #4016, it was asked to find the maximum value of the given expression while in #4068, it becomes a proof question with the maximum value given. So, it can not be viewed as a "variation". Two different solutions to #4016 given by Arkady Alt and Šefket Arslanagić have appeared on pp. 85-86 of ***Crux*** 42 (2). It is interesting to note that both of them are also among the 25 solvers to the current problem but neither made any reference to #4016 !

4069. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let $(u_n)_{n\geq 0}$ be an arithmetic progression with a positive common difference d and with $u_1 > 0$. Let $(x_n)_{n\geq 0}$ be a sequence with $x_0 = 0, x_1 = x_2 = 1$ and

$$\sum_{k=1}^{n} u_k x_k = u_n x_{n+2} + d(x_4 - x_{n+3}) - x_2 u_1, \ \forall n \geq 0.$$

Prove that $(x_n)_{n\geq 0}$ is the Fibonacci sequence.

There were twelve correct solutions. All were variants of the ones below, with three using the closed form of the Fibonacci sums in Solution 2.

Solution 1.

When $n = 0$, the condition is that

$$0 = u_0 x_2 + d(x_4 - x_3) - u_1 x_2 = d(x_4 - x_3 - x_2),$$

whence $x_4 = x_3 + x_2$. When $n = 1$, we have that $u_1 x_1 = u_1 x_3 - u_1 x_2$, whence $x_3 = x_1 + x_2$. Since $x_0 = 0 = F_0$ and $x_1 = x_2 = 1 = F_1 = F_2$, then $x_3 = F_3$ and $x_4 = F_4$.

For $n \geq 1$, we have that

$$\begin{aligned} u_{n+1}x_{n+1} &= [u_{n+1}x_{n+3} + d(x_4 - x_{n+4}) - u_1 x_2] - [u_n x_{n+2} + d(x_4 - x_{n+3}) - u_1 x_2] \\ &= -dx_{n+4} + (u_{n+1} + d)x_{n+3} - (u_{n+1} - d)x_{n+2}, \end{aligned}$$

so that

$$dx_{n+4} = d(x_{n+3} + x_{n+2}) + u_{n+1}(x_{n+3} - x_{n+2} - x_{n+1}).$$

We establish the result by induction. Suppose that $x_k = F_k$ for $0 \leq k \leq n+3$. This is true for $n = 1$. The foregoing equation establishes that if $x_k = F_k$ for $k = n+1, n+2, n+3$, then $dx_{n+4} = dF_{n+4} + u_{n+1}(0)$ and $x_{n+4} = F_{n+4}$.

Solution 2.

The following Fibonacci relationships are easily established by induction for $n \geq 1$:

$$F_1 + F_2, + \cdots + F_n = F_{n+2} - 1 \quad \text{and} \quad \sum_{k=1}^{n} (k-1)F_k = (n-1)F_{n+2} - F_{n+3} + 3.$$

As in Solution 1, we show that $x_k = F_k$ for $0 \leq k \leq 4$. Suppose, as an induction hypothesis, this holds for $1 \leq k \leq n+2$. By the foregoing relationships, we have that

$$\begin{aligned} \sum_{k=1}^{n} u_k x_k &= \sum_{k=1}^{n} [u_1 + (k-1)d]F_k \\ &= u_1 \sum_{k=1}^{n} F_k + d \sum_{k=1}^{n} (k-1)F_k \\ &= u_1[F_{n+2} - 1] + d(n-1)F_{n+2} - dF_{n+3} + 3d \\ &= F_{n+2}[u_1 + (n-1)d] + d(3 - F_{n+3}) - u_1 \\ &= u_n F_{n+2} + d(F_4 - F_{n+3}) - u_1 F_2. \end{aligned}$$

However, the given condition provides that

$$\sum_{k=1}^{n} u_k x_k = u_n F_{n+2} + d(F_4 - x_{n+3}) - u_1 F_2.$$

Therefore $x_{n+3} = F_{n+3}$, and the result holds.

4070. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Compute

$$\lim_{n\to\infty}\left[\frac{1}{\ln n}\left(\frac{\arctan 1}{n}+\frac{\arctan 2}{n-1}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right)\right].$$

We received six correct and complete solutions. We present the solution by Joel Schlosberg.

Let $H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. Since $\arctan x$ is an increasing function,

$$\begin{aligned}&\frac{1}{\ln n}\left[\frac{\arctan 1}{n}+\frac{\arctan 2}{n-1}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right]\\ &\le \frac{1}{\ln n}\left(\frac{\arctan n}{n}+\frac{\arctan n}{n-1}+\cdots+\frac{\arctan n}{2}+\arctan n\right)\\ &=\frac{H_n}{\ln n}\cdot\arctan n;\end{aligned}$$

and for any positive integer m, if $n\ge m$,

$$\begin{aligned}&\frac{1}{\ln n}\left[\frac{\arctan 1}{n}+\frac{\arctan 2}{n-1}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right]\\ &\ge \frac{1}{\ln n}\left(\frac{\arctan m}{n-m+1}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right)\\ &\ge \frac{1}{\ln n}\left(\frac{\arctan m}{n-m+1}+\cdots+\frac{\arctan m}{2}+\arctan m\right)\\ &=\frac{H_{n-m+1}}{\ln n}\cdot\arctan m.\end{aligned}$$

It is well known that H_n is asymptotic to $\ln n$ and $\lim_{x\to\infty}\arctan x=\pi/2$. Therefore,

$$\lim_{n\to\infty}\frac{H_n}{\ln n}\cdot\arctan n=\frac{\pi}{2}$$

and

$$\begin{aligned}&\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{H_{n-m+1}}{\ln n}\cdot\arctan m\right)\\ &=\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{\ln(n-m+1)}{\ln n}\cdot\arctan m\right)=\lim_{m\to\infty}\arctan m=\frac{\pi}{2}\end{aligned}$$

so by the squeeze theorem,

$$\lim_{n\to\infty}\frac{1}{\ln n}\left(\frac{\arctan 1}{n}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right)=\frac{\pi}{2}.$$