

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(5), p. 197–198.

OC231. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x)f(y) = f(x+y) + xy. \quad (1)$$

for all $x, y \in \mathbb{R}$.

Originally problem 4 of the 2014 Balkan Mathematical Olympiad TST.

We received 8 correct submissions and 1 incorrect submission. We present the solution by Elnaz Hessami Pilehrood.

Substituting $x = 0$ in this equation, we get

$$f(0)f(y) = f(y) \quad \text{or} \quad f(y)(f(0) - 1) = 0.$$

Therefore, either $f(y) = 0$ or $f(0) = 1$.

We can see that $f = 0$ does not satisfy (1), so $f = 0$ cannot be such a function. Therefore, $f(0) = 1$.

When $f(0) = 1$, we can substitute $x = 1$ and $y = -1$ in (1) to get

$$f(1)f(-1) = f(0) - 1 = 0$$

and therefore, $f(1) = 0$ or $f(-1) = 0$.

If $f(1) = 0$, substitute $y = 1$ to get

$$f(x)f(1) = f(x+1) + x \quad \text{or} \quad 0 = f(x+1) + x,$$

which implies $f(x+1) = -x$ or $f(x) = 1 - x$. The function $f(x) = 1 - x$ satisfies all conditions, as $(1-x)(1-y) = 1 - x - y + xy$. If $f(-1) = 0$, substitute $y = -1$ to get

$$f(x)f(-1) = f(x-1) - x \quad \text{or} \quad f(x-1) = x.$$

The function $f(x) = x+1$ satisfies all conditions, as $(1+x)(1+y) = 1 + x + y + xy$.

Therefore, all such functions are $f(x) = 1 - x$ and $f(x) = 1 + x$.

OC232. Given a positive integer m , Prove that there exists a positive integers n_0 such that all first digits after the decimal points of $\sqrt{n^2 + 817n + m}$ in decimal representation are equal, for all integers $n > n_0$.

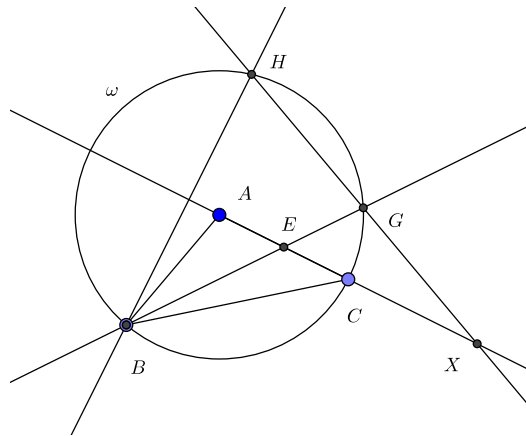
Originally problem 5 from day 2 of the 2014 China Western Mathematical Olympiad.

No submitted solutions.

OC233. Let ω be a circle with center A and radius R . On the circumference of ω four distinct points B, C, G, H are taken in that order in such a way that G lies on the extended B -median of the triangle ABC , and H lies on the extension of altitude of ABC from B . Let X be the intersection of the straight lines AC and GH . Show that the segment AX has length $2R$.

Originally problem 4 of the 2014 Italy Mathematical Olympiad.

We received 8 correct submissions. We present the solution by Somasundaram Muralidharan.



There is no loss of generality in assuming that the circle ω is $|z| = 1$ in the complex plane. Let the B, C, G, H be represented by the complex numbers b, c, g, h respectively. The slope of BG is given by $\frac{g-b}{\bar{g}-\bar{b}} = -bg$. The midpoint E of AC is $\frac{c}{2}$. Since G lies on the extension of BE , slope of BE must be equal to the slope of BG . Hence, we have

$$-bg = \frac{b - \frac{c}{2}}{\bar{b} - \frac{\bar{c}}{2}} = \frac{\frac{2b-c}{2}}{\frac{1}{\bar{b}} - \frac{1}{2\bar{c}}} = bc \left(\frac{2b-c}{2c-b} \right).$$

Thus,

$$g = -c \left(\frac{2b-c}{2c-b} \right).$$

Now the slope of AC is $\frac{c}{\bar{c}} = c^2$ and since BH is perpendicular to AC , slope of BH is $-c^2$. Since the slope of BH is also given by $\frac{b-h}{\bar{b}-\bar{h}} = -bh$, we obtain $-c^2 = -bh$ or $h = \frac{c^2}{b}$.

Now, the equation of AC is $Z = c^2\bar{Z}$ and that of HG is $Z - h = -hg(\bar{Z} - \bar{h})$. The lines HG and AC meet at X . Solving for \bar{Z} , we obtain \bar{x} , the conjugate of the

complex number x representing X . Thus,

$$\bar{x} = \frac{h+g}{c^2+hg} = \frac{\frac{c^2}{b} - c\left(\frac{2b-c}{2c-b}\right)}{c^2 - \frac{c^3}{b}\left(\frac{2b-c}{2c-b}\right)} = \frac{2c^2 - bc - 2b^2 + bc}{bc(2c-b) - c^2(2b-c)} = \frac{2(c^2 - b^2)}{c(c^2 - b^2)} = 2\bar{c}.$$

Hence, $x = 2c$ and $AX = 2AC$. Thus, AX is twice the radius of ω .

OC234. Let N be an integer, $N > 2$. Arnold and Bernold play the following game: there are initially N tokens on a pile. Arnold plays first and removes k tokens from the pile, $1 \leq k < N$. Then Bernold removes m tokens from the pile, $1 \leq m \leq 2k$ and so on, that is, each player, on its turn, removes a number of tokens from the pile that is between 1 and twice the number of tokens his opponent took last. The player that removes the last token wins.

For each value of N , find which player has a winning strategy and describe it.

Originally problem 3 from day 1 of the 2014 Brazil National Olympiad.

No submitted solutions.

OC235. Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a+i, b+j) > 1$ for all $i, j \in \{0, 1, \dots, n\}$, then

$$\min\{a, b\} > c^n \cdot n^{\frac{n}{2}}.$$

Originally problem 6 from day 2 of the 2014 USA Mathematical Olympiad.

We present the solution by Oliver Geupel. There were no other submissions.

We prove the stronger bound

$$\min\{a, b\} > (cn)^n.$$

For $i, j \in \{0, 1, \dots, n\}$, let p_{ij} be a prime such that $p_{ij} | \gcd(a+i, b+j)$. Then, for every prime number $p \in \{p_{ij} \mid i, j \in \{0, 1, \dots, n\}\}$, the total number of pairs (i, j) such that $p = p_{ij}$, is not greater than $\left\lceil \frac{n+1}{p} \right\rceil^2 < \left(\frac{n+1}{p} + 1\right)^2$. Let n be a large integer and $N = (n+1)^2$. Then, the total number of pairs (i, j) such that $p_{ij} \leq N$, is not greater than

$$\sum_{p \leq N} \left(\frac{n+1}{p} + 1\right)^2 < (n+1)^2 \sum_{p \text{ prime}} \frac{1}{p^2} + 2(n+1) \sum_{p \leq N} \frac{1}{p} + \sum_{p \leq N} 1.$$

It is known that

$$\sum_{p \text{ prime}} \frac{1}{p^2} = 0.45\dots < \frac{1}{2} \quad \text{and} \quad \sum_{p \leq N} \frac{1}{p} = O(\log \log N),$$

see [1] and [2], respectively. Also,

$$\sum_{p \leq N} 1 = O\left(\frac{N}{\log N}\right)$$

by the prime number theorem. Therefore, for every sufficiently large n , the total number of pairs (i, j) such that $p_{ij} \leq N$, is less than $(n+1)^2/2$.

By the pigeonhole principle there is an index $i \in \{0, 1, \dots, n\}$ such that for more than half of the numbers $j \in \{0, 1, \dots, n\}$ the respective prime p_{ij} is greater than N . Let i_0 denote such an index i . The primes p_{i_0j} that are greater than N are distinct and, therefore, are coprime divisors of $a + i_0$. A similar argument holds for the number b . We deduce that $\min\{a, b\} \geq N^{(n+1)/2} - n > n^n$ for every n exceeding some bound n_0 . Let $c = 1/n_0$. For $n < n_0$ we have $\min\{a, b\} \geq 1 > (cn)^n$. For $n \geq n_0$ we obtain $\min\{a, b\} > n^n \geq (cn)^n$. Hence the result.

References

- [1] The on-line encyclopedia of integer sequences, published electronically at <https://oeis.org>, sequence A085548.
- [2] Tom M. Apostol, Introduction to analytic number theory, Springer 2010, theorem 4.12, page 90.

