FOCUS ON...

No. 23

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Vieta's Formulas

Introduction

For $k = 1, 2, \ldots, n$, the kth elementary symmetric polynomial $e_k(X_1, X_2, \ldots, X_n)$ in the indeterminates $X_1, X_2, \ldots, X_n$ is defined by

$$e_k(X_1, X_2, \ldots, X_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} X_{i_1} X_{i_2} \cdots X_{i_k},$$

a sum of $\binom{n}{k}$ terms.

In particular, $e_1(X_1, X_2, \ldots, X_n) = X_1 + X_2 + \cdots + X_n$ and $e_n(X_1, X_2, \ldots, X_n) = X_1 X_2 \cdots X_n$.

If $P = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$ is a polynomial with complex coefficients and degree n ($a_n \neq 0$), Vieta's formulas establish a link between the n roots $z_1, z_2, \ldots, z_n$ of P (counted with multiplicity) and the coefficients of P. They can be summarized as follows

$$e_k(z_1, z_2, \ldots, z_n) = (-1)^k \frac{a_{n-k}}{a_n} \qquad (k = 1, 2, \ldots, n),$$

and are easily deduced by expanding $P = a_n(X - z_1)(X - z_2) \cdots (X - z_n)$.

Note that $e_1(z_1, z_2, \ldots, z_n) = -\frac{a_{n-1}}{a_n}$ and $e_n(z_1, z_2, \ldots, z_n) = (-1)^n \frac{a_0}{a_n}$ generalize the familiar formulas obtained when the degree of P is 2.

We will show these formulas at work first in a few algebraic problems and, in a second part, in combination with inequalities.

Three algebraic problems

Our first problem was set at the 1997 Mathematical Olympiad in Bosnia and Hercegovina [2000 : 325-6 ; 2002 : 485]:

> Solve the system of equations in $\mathbb{R}^3$:
>
> $$8(x^3 + y^3 + z^3) = 73, \qquad 2(x^2 + y^2 + z^2) = 3(xy + yz + zx), \qquad xyz = 1.$$

Here is a variant of the featured solution: suppose that (z_1, z_2, z_3) is a solution. Then z_1, z_2, z_3 are the roots of the polynomial $P = X^3 - aX^2 + bX - 1$, where $a = z_1 + z_2 + z_3$ and $b = z_1 z_2 + z_2 z_3 + z_3 z_1$. Adding the three relations

$$z_k^3 - a z_k^2 + b z_k - 1 = 0, \quad k = 1, 2, 3,$$

we obtain

$$\frac{73}{8} - a \cdot \left(\frac{3b}{2}\right) + ba - 3 = 0,$$

so that $ab = \frac{49}{4}$. Since $a^2 - 2b = z_1^2 + z_2^2 + z_3^2 = \frac{3b}{2}$, we have $a^2 = \frac{7b}{2}$, hence $a^3 = \frac{7ab}{2} = \frac{7^3}{2^3}$. We deduce $a = b = \frac{7}{2}$ and

$$P = X^3 - \frac{7}{2}X^2 + \frac{7}{2}X - 1 = (X-1)(X-2)(X-\frac{1}{2}).$$

Thus, (z_1, z_2, z_3) is a permutation of $(1, 2, \frac{1}{2})$. Conversely, it is readily checked that each of the six permutations of $(1, 2, \frac{1}{2})$ is a solution.

We now consider an example involving a polynomial of degree 4 :

> Suppose that a and b are two of the roots of the polynomial X^4+X^3-1. Find a polynomial of which ab is a root.

Let c, d denote the complex roots other than a, b of $X^4 + X^3 - 1$. Vieta's formulas give:

$$a+b+c+d = -1,\ ab+ac+ad+bc+bd+cd = 0,\ abc+abd+bcd+acd = 0,\ abcd = -1,$$

or, adopting the notations $p = a + b, q = ab, r = c + d, s = cd$,

$$p + r = -1, \quad q + s + pr = 0, \quad qr + sp = 0, \quad qs = -1.$$

Note that $q \neq 0$ since 0 is not a root of $X^4 + X^3 - 1$. The elimination of p, r, s will provide a condition on q. Specifically, substituting $s = -\frac{1}{q}$ and $r = -1 - p$ in the two central relations, we obtain conditions on p and q, namely:

$$q^2 - 1 - pq(p+1) = 0, \qquad p + q^2(p+1) = 0.$$

Lastly, substituting $p = -\dfrac{q^2}{q^2+1}$ (obtained from the latter) in the first equality yields, after some algebra, $q^6 + q^4 + q^3 - q^2 - 1 = 0$ so that ab is a root of $X^6 + X^4 + X^3 - X^2 - 1$.

Vieta's formulas alone cannot provide the roots of the polynomial. However, accompanied with some extra information, they can be used in view of determining the roots. We give an example with a polynomial of degree 5 :

> Find the roots of $P = X^5 - 4X^4 + 9X^3 - 21X^2 + 20X - 5$, given that the product of two of the roots is 5.

Vieta's formulas are a bit more complicated in the case of the fifth degree than with polynomial of degree 3 or 4. However, if we denote by z_1, z_2, z_3, z_4, z_5 the roots of P and suppose without loss of generality that $z_1 z_2 = 5$, we are led to the following simplified form of Vieta's formulas:

$$\begin{aligned}
&z_3z_4z_5 = 1 \text{ (from } \ z_1z_2z_3z_4z_5 = 5),\\
&z_1 + z_2 + z_3 + z_4 + z_5 = 4,\\
&(z_1 + z_2)(z_3 + z_4 + z_5) + (z_3z_4 + z_4z_5 + z_3z_5) = 4,\\
&5(z_3 + z_4 + z_5) + (z_1 + z_2)(z_3z_4 + z_4z_5 + z_3z_5) = 20,\\
&z_1 + z_2 + 5(z_3z_4 + z_4z_5 + z_3z_5) = 20,
\end{aligned}$$

a clear invitation to set

$$a = z_1 + z_2, \ b = z_3z_4 + z_4z_5 + z_3z_5, \ c = z_3 + z_4 + z_5.$$

These numbers satisfy

$$a + c = 4, \ ac + b = 4, \ 5c + ab = 20, \ a + 5b = 20,$$

from which we successively deduce

$$b = 4 - a(4 - a) = a^2 - 4a + 4 = (a - 2)^2 \quad \text{and} \quad a + 5(a - 2)^2 = 20,$$

hence $a = 0$ or $a = \frac{19}{5}$. But the latter leads to $c = \frac{1}{5}$ (from $a + c = 4$), giving $b = 5$ (with $5c + ab = 20$), in contradiction with $b = (a - 2)^2 = \frac{81}{25}$. Thus, we must have $a = 0$ and then $b = 4 = c$.

To complete the solution, it just remains to remark that $z_1z_2 = 5$ and $z_1 + z_2 = 0$ yield the roots $i\sqrt{5}$ and $-i\sqrt{5}$, while z_3, z_4, z_5 are the roots of $X^3 - 4X^2 + 4X - 1$ (since $z_3 + z_4 + z_5 = 4, z_3z_4 + z_4z_5 + z_3z_5 = 4$ and $z_3z_4z_5 = 1$). The factorization

$$X^3 - 4X^2 + 4X - 1 = (X - 1)(X^2 - 3X + 1)$$

gives the three missing roots: $1, \frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$.

Vieta's formulas and inequalities

Our first example, problem 766 of *The College Mathematics Journal*, offers a necessary condition for the roots of a polynomial of degree 3 in $\mathbb{R}[x]$ to be real and nonnegative.

> Suppose that the polynomial with real coefficients $A(z) = a_0 + a_1z + a_2z^2 + z^3$ has all its zeros real and nonnegative. Prove that
>
> $$9a_0^2 + a_1^2a_2^2 \geq \frac{4}{3}a_1^3 + 6a_0a_1a_2.$$

Let x_1, x_2, x_3 denote the nonnegative zeros of $A(z)$, so that

$$a_2 = -(x_1 + x_2 + x_3), \quad a_1 = x_1x_2 + x_2x_3 + x_3x_1, \quad a_0 = -x_1x_2x_3.$$

The result is obvious if $x_1 = x_2 = x_3 = 0$. Otherwise, we have $x_1 + x_2 + x_3 > 0$ and, observing that $9a_0^2 + a_1^2a_2^2 - \frac{4}{3}a_1^3 - 6a_0a_1a_2$ is a homogeneous polynomial in

x_1, x_2, x_3, we may even suppose $x_1 + x_2 + x_3 = 1$. We are reduced to proving $9a_0^2 + a_1^2 - \frac{4}{3}a_1^3 + 6a_0a_1 \geq 0$ or

$$H^2 + 1 - \frac{4}{3}a_1 - 2H \geq 0 \tag{*}$$

if we set $H = 3 \cdot \dfrac{x_1x_2x_3}{x_1x_2 + x_2x_3 + x_3x_1} \left(= \dfrac{-3a_0}{a_1} \right)$. Note that H is the harmonic mean of x_1, x_2, x_3.

The well-known inequality $x_1^2 + x_2^2 + x_3^2 \geq x_1x_2 + x_2x_3 + x_3x_1$ gives

$$1 = (x_1 + x_2 + x_3)^2 \geq 3(x_1x_2 + x_2x_3 + x_3x_1)$$

that is, $a_1 \leq \frac{1}{3}$, and therefore

$$H^2 + 1 - \frac{4}{3}a_1 - 2H \geq H^2 - 2H + \frac{5}{9} = \left(H - \frac{5}{3}\right)\left(H - \frac{1}{3}\right).$$

Now, $(*)$ follows from $H \leq \dfrac{x_1 + x_2 + x_3}{3} = \dfrac{1}{3}$.

In our next problem, again from *The College Mathematics Journal* (No 879), the advanced reader will recognize the Maclaurin inequalities: with the notation of our first part, if $z_1, z_2, \ldots, z_n$ are positive real numbers, then for $k = 1, 2, \ldots, n-1$,

$$\left(\frac{e_k(z_1, \ldots, z_n)}{\binom{n}{k}}\right)^{1/k} \geq \left(\frac{e_{k+1}(z_1, \ldots, z_n)}{\binom{n}{k+1}}\right)^{1/(k+1)},$$

a chain of inequalities from the arithmetic mean of $z_1, z_2, \ldots, z_n$ to their geometric mean.

The problem is stated as follows:

> Consider the polynomial $f(x) = x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4$, where a, b, c, and d are positive real numbers. Prove that if f has four positive distinct roots, then $a > b > c > d$.

Of course, the Maclaurin inequalities quickly give the answer. However the following direct solution offers a hint towards a general proof of the Maclaurin inequalities [for such a proof we refer the reader to **[1]** or **[2]**] .

Let x_1, x_2, x_3, x_4 be the four positive distinct roots of f with, $x_1 < x_2 < x_3 < x_4$. From Rolle's Theorem, the derivative $f'(x) = 4x^3 - 12ax^2 + 12b^2x - 4c^3$ has three positive distinct roots y_1, y_2, y_3 (with $x_1 < y_1 < x_2 < y_2 < x_3 < y_3 < x_4$) and the second derivative $f''(x) = 12x^2 - 24ax + 12b^2$ has two positive distinct roots z_1, z_2.

Now, using Vieta's formulas and the arithmetic mean - geometric mean inequality, we successively obtain

$$a = \frac{z_1 + z_2}{2} > \sqrt{z_1 z_2} = b,$$
$$b = \left(\frac{y_1 y_2 + y_2 y_3 + y_3 y_1}{3}\right)^{1/2} > \left(\sqrt[3]{y_1^2 y_2^2 y_3^2}\right)^{1/2} = \sqrt[3]{y_1 y_2 y_3} = c,$$
$$c = \left(\frac{x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4}{4}\right)^{\frac{1}{3}} > \left(\sqrt[4]{x_1^3 x_2^3 x_3^3 x_4^3}\right)^{\frac{1}{3}} = \sqrt[4]{x_1 x_2 x_3 x_4} = d,$$

so that $a > b > c > d$.

We complete this number with a couple of exercises.

Exercises

1. Given that the polynomial $X^3 - 5X + m$ has two roots z_1, z_2 such that $z_1 + z_2 = 2z_1 z_2$, find the value of m and all the roots.

2. Let $Q(x) \in \mathbb{R}[x]$ and $P(x) = a + bx + cx^2 + x^3 Q(x)$ where a, b, c are real numbers and $ac \neq 0$. Prove that if all the roots of P are real, then $b^2 > 2ac$. (Hint: if n is the degree of P, consider $x^n P(1/x)$.)

References

[1] R. Mosier, J. Nieto, Solution to problem 879, *The College Mathematics Journal*, Vol. 40, No 3, May 2009, p. 219.

[2] J. M. Steele, *The Cauchy-Schwarz Master Class*, Cambridge U. Press, 2006, chapter 12.