OLYMPIAD SOLUTIONS


OC226. In a triangle $ABC$, let $D$ be the point on the segment $BC$ such that $AB + BD = AC + CD$. Suppose that the points $B$, $C$ and the centroids of triangles $ABD$ and $ACD$ lie on a circle. Prove that $AB = AC$.

Originally problem 1 of the 2014 India National Olympiad.

We received six correct submissions. We present the solution by Titu Zvonaru (similar to Šefket Arslanagić).

As usual, let $a, b, c$ be the sides of $\triangle ABC$ and let $2s = a + b + c$. Let $T$ be the midpoint of $AD$ and let $G_1$ and $G_2$ be the centroids of triangles $ABD$ and $ACD$ respectively. Since $BD - CD = b - c$ and $BD + CD = a$, we see that $BD = s - c$ and $CD = s - b$. Now, $BG_1G_2C$ is cyclic if and only if $TG_1 \cdot TB = TG_2 \cdot TC$. Since medians are divided by the centroid in a $2 : 1$ ratio, we see this holds if and only if $\frac{TB^2}{3} = \frac{TC^2}{3}$. This is true if and only if $4TB^2 = 4TC^2$ which, by the formula for a median’s length, holds if and only if

$$2BD^2 + 2BA^2 - AD^2 = 2CD^2 + 2CA^2 - AD^2.$$ 

Substituting the values from before, this holds if and only if $(s - c)^2 + c^2 = (s - b)^2 + b^2$ which is equivalent to $(b - c)(b + c - a) = 0$. Since $a \neq b + c$ by the triangle inequality, we see that $b = c$ and hence $AB = AC$.

OC227. In a bag there are 1007 black and 1007 white balls, which are randomly numbered 1 to 2014. In every step we draw one ball and put it on the table; also if we want to, we may choose two different colored balls from the table and put them in a different bag. If we do that we earn points equal to the absolute value of their differences. How many points can we guarantee to earn after 2014 steps?

Originally problem 1 from day 1 of the 2014 Turkey Mathematical Olympiad.

No submitted solutions.

OC228. Let $k$ be a nonzero natural number and $m$ an odd natural number. Prove that there exist a natural number $n$ such that the number $m^n + n^m$ has at least $k$ distinct prime factors.

Originally problem 4 from day 1 of the 2014 Romanian Team Selection Test.

No submitted solutions.

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OC229. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $x, y \in \mathbb{R}^+$,

$$f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) = f(y)$$

*Originally problem 4 from day 2 of the 2014 Iran Team Selection Test.*

We received two correct submissions. We present the solution by Oliver Geupel.

The function

$$f(x) = \frac{1}{x}$$

is a solution because, for $x, y \in \mathbb{R}^+$,

$$f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) = \frac{1}{(x+1)y} + \frac{x}{(x+1)y} = \frac{1}{y} = f(y).$$

We show that there is no other solution.

Suppose $f$ is any solution of the problem.

To begin with, we prove that

$$f(x) \leq \frac{1}{x}$$  \hspace{1cm} (1)

for every $x > 0$. Suppose that, contrary to our claim, $f(a) > \frac{1}{a}$ for some $a > 0$.

Putting

$$x = \frac{1}{af(a)-1}, \quad y = a,$$

we obtain

$$\frac{x+1}{xf(y)} = y,$$

that is,

$$f\left(\frac{y}{f(x+1)}\right) = f(y) - f\left(\frac{x+1}{xf(y)}\right) = 0,$$

a contradiction. This proves (1) for $x > 0$.

Next we show that, for every $x \geq 1$,

$$f(x) = \frac{1}{x}.$$  \hspace{1cm} (2)

By (1), we have for all $x, y \in \mathbb{R}^+$,

$$f(y) = f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) \leq \frac{f(x+1)}{y} + \frac{x}{x+1} \cdot f(y),$$

so that $yf(y) \leq (x+1)f(x+1)$. It follows that $xf(x)$ is identically constant for $x > 1$, say $xf(x) = c \leq 1$. Moreover, for all $x > 1$, we have

$$\frac{x}{f(x+1)} = \frac{x(x+1)}{c} > 1, \quad \frac{x+1}{xf(x)} = \frac{x+1}{c} > 1,$$
so that
\[ \frac{c^2}{x} = \frac{c^2}{(x+1)x} + \frac{c^2x}{(x+1)x} = f \left( \frac{x}{f(x+1)} \right) + f \left( \frac{x+1}{xf(x)} \right) = f(x) = \frac{c}{x}, \]
which implies \( c = 1 \). We obtain (2) for every \( x > 1 \).

By (1), we have \( \frac{2}{f(1)} \geq 2 \), whence
\[ f(1) = f \left( \frac{1}{f(2)} \right) + f \left( \frac{2}{f(1)} \right) = \frac{1}{2} + \frac{1}{2} f(1), \]
that is, \( f(1) = 1 \). This proves that (2) holds for every \( x \geq 1 \).

Finally, let \( P(n) \) denote the assertion that equation (2) is true for every \( x \geq 2^{-n} \). It is enough to prove \( P(n) \) for all nonnegative integers \( n \). We do so by mathematical induction. We have already established the base case \( n = 0 \). For the induction step suppose \( P(n) \). Let
\[ \frac{1}{2^{n+1}} \leq y < \frac{1}{2^n}, \quad x = \frac{2^n y}{1-2^n y}. \]
Then,
\[ f(x+1) = \frac{1}{x+1} = 1 - 2^ny \]
and
\[ \frac{y}{f(x+1)} = \frac{y}{1-2^ny} \geq \frac{1}{2^n}. \]
By (1),
\[ \frac{x+1}{xf(y)} \geq \frac{x+1}{x} \cdot y = \frac{1}{2^n}. \]
By induction we deduce
\[ f(y) = f \left( \frac{y}{f(x+1)} \right) + f \left( \frac{x+1}{xf(y)} \right) = \frac{1}{(x+1)y} + \frac{x}{x+1} \cdot f(y) \]
and conclude
\[ f(y) = \frac{1}{y}, \]
which proves \( P(n+1) \) and the proof is complete.

**OC230.** Find, with justification, all positive real numbers \( a, b, c \) satisfying the system of equations :
\[ a\sqrt{b} = a + c, \quad b\sqrt{c} = b + a, \quad c\sqrt{a} = c + b. \]

Originally problem 2 of the 2014 Singapore Senior Math Olympiad.

We received seven correct submissions. We present the solution by Albert Stadler.
We claim that \( a = b = c = 4 \) is the only solution in positive numbers \( a, b, c \). It is easy to verify that this is a solution so we now show this is the only one. Let \( a = u^2, b = v^2 \) and \( c = w^2 \) in order to eliminate square roots. With these new variables, we need to show that \( u = v = w = 2 \) is the only positive solution of

\[
\begin{align*}
    u^2 v &= u^2 + w^2 \\
v^2 w &= v^2 + u^2 \\
w^2 u &= w^2 + v^2
\end{align*}
\]

Using the AM-GM inequality, we see that

\[
u^2 v = u^2 + w^2 \geq 2uw
\]

implying that \( uv \geq 2w \). Similarly, \( vw \geq 2u \) and \( uw \geq 2v \). Thus, multiplying the first two inequalities gives

\[
(uv)(vw) \geq 4uvw
\]

implying that \( u^2 \geq 4 \) and hence \( u \geq 2 \) (recall we only want positive solutions). Similarly, \( u \geq 2 \) and \( w \geq 2 \). Then, the triple of equalities above implies that

\[
\begin{align*}
v - 1 &= \left( \frac{w}{u} \right)^2 \\
w - 1 &= \left( \frac{u}{v} \right)^2 \\
u - 1 &= \left( \frac{v}{w} \right)^2
\end{align*}
\]

and hence, multiplying these together yields \( (u - 1)(v - 1)(w - 1) = 1 \). However, \( u \geq 2, v \geq 2 \) and \( w \geq 2 \). Hence, \( u = v = w = 2 \).