No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


4041. Proposed by Arkady Alt.

Let $a, b$ and $c$ be the side lengths of a triangle $ABC$. Let $AA', BB'$ and $CC'$ be the heights of the triangle and let $a_p = B'C', b_p = C'A'$ and $c_p = A'B'$ be the sides of the orthic triangle. Prove that:

a) $a^2 (b_p + c_p) + b^2 (c_p + a_p) + c^2 (a_p + b_p) = 3abc$;

b) $a_p + b_p + c_p \leq s$, where $s$ is the semiperimeter of $ABC$.

We received 15 correct solutions and present the solution by Michel Bataille.

We show (a) and (b) in the case when $\Delta ABC$ has no obtuse angle and provide a counter-example in the opposite case.

First, suppose that angles $A, B, C$ are acute. Since $\Delta AB'B$ is right-angled with $\angle AB'B = 90^\circ$, we have $AB' = a \cos A$. Similarly, $AC' = b \cos A$, and it follows that

$$B'C'^2 = c^2 \cos^2 A + b^2 \cos^2 A - 2bc \cos^3 A = (c^2 + b^2 - 2bc \cos A) \cos A = a^2 \cos^2 A$$

and so $a_p = B'C' = a \cos A$. In a similar way, we obtain $b_p = A'C' = b \cos B$ and $c_p = A'B' = c \cos C$.

Now we calculate $X = a^2 (b_p + c_p) + b^2 (c_p + a_p) + c^2 (a_p + b_p)$ as follows:

$$X = a^2 b \cos B + a^2 c \cos C + b^2 c \cos C + b^2 a \cos A + bc^2 \cos B + c^2 a \cos A$$

$$= ab(a \cos B + b \cos A) + bc(b \cos C + c \cos B) + ca(c \cos A + a \cos C)$$

$$= abc + bca + cab = 3abc,$$

as desired. Denoting by $r$ and $R$ the inradius and the circumradius of $\Delta ABC$ and using the Law of Sines, we get

$$a_p + b_p + c_p = a \cos A + b \cos B + c \cos C$$

$$= R(\sin 2A + \sin 2B + \sin 2C)$$

$$= 4R \sin A \sin B \sin C$$

$$= 4R \cdot \frac{abc}{8R^3} = \frac{4rRs}{2R^2} = s \cdot \frac{2r}{R},$$

and the result $a_p + b_p + c_p \leq s$ follows from Euler’s inequality $2r \leq R$. 

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If \( \triangle ABC \) is right-angled, say \( \angle BAC = 90^\circ \), results (a) and (b) continue to hold if we take, as is natural, \( a_p = 0 \), \( b_p = c_p = h \), where \( h = AA' \). Indeed, we have
\[
3abc = 3a \cdot ah = 3a^2 h
\]
and
\[
a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p) = a^2 \cdot 2h + b^2 \cdot h + c^2 \cdot h
\]
\[
= h(b^2 + c^2 + 2a^2) = 3a^2 h.
\]

Also, the inequality \( a_p + b_p + c_p \leq s \) rewrites as
\[
4h \leq a + b + c
\]
or
\[
4bc \leq a^2 + a(b + c).
\]

Since \( b + c \geq 2 \sqrt{bc} \) and \( a^2 = b^2 + c^2 \geq 2bc \), we have
\[
a^2 + a(b + c) \geq 2bc + 2 \sqrt{2bc} \cdot 2 \sqrt{bc} = (2 + 2 \sqrt{2})bc \geq 4bc.
\]

None of these results is correct, however, if an angle of \( \triangle ABC \) is obtuse, as the following example shows. Consider a triangle \( ABC \) with \( \angle BAC = 120^\circ \) and \( AB = AC \). Then \( b = c \), \( a = c \sqrt{3} \), and \( a_p = b_p = c_p = \frac{a}{2} = \frac{c \sqrt{3}}{2} \). One easily finds that \( 3abc = 3c^3 \sqrt{3} \), while
\[
a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p) = 5c^3 \sqrt{3}.
\]
Also,
\[
a_p + b_p + c_p = \frac{3c \sqrt{3}}{2} > \frac{(2 + \sqrt{3})c}{2} = s.
\]

4042. Proposed by Leonard Giugiuc and Diana Trailescu.

Let \( a, b \) and \( c \) be real numbers in \([0, \pi/2]\) such that \( a + b + c = \pi \). Prove the inequality
\[
2 \sqrt{2} \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2} \geq \sqrt{\cos a \cos b \cos c}.
\]

We received 14 correct solutions. We present the solution by Scott Brown. Similar solutions came from Arslanagić Sefket, Michel Bataille, Andrea Fanchini, and John Hewel.

In [1] and [2] respectively, we find the identities
\[
\sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2} = \frac{r}{4R}
\]
and
\[
\cos a \cos b \cos c = \frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2},
\]
where \( R, r, \) and \( s \) are the circumradius, inradius, and semiperimeter of the triangle. We square both sides of the original inequality to obtain the equivalent statement
\[
8 \sin^2 \frac{a}{2} \sin^2 \frac{b}{2} \sin^2 \frac{c}{2} \leq \cos a \cos b \cos c,
\]

Crux Mathematicorum, Vol. 42(5), May 2016
into which we substitute the identities (1) and (2). The resulting inequality is equivalent to one due to Gerretsen [3]:

\[ s^2 \leq 4R^2 + 4Rr + 3r^2. \]

References


4043. Proposed by Michel Bataille.

Suppose that the lines \( m \) and \( n \) intersect at \( A \) and are not perpendicular. Let \( B \) be a point on \( n \), with \( B \neq A \). If \( F \) is a point of \( m \), distinct from \( A \), show that there exists a unique conic \( C_F \) with focus \( F \) and focal axis \( BF \), intersecting \( n \) orthogonally at \( A \). Given \( \epsilon > 0 \), how many of the conics \( C_F \) have eccentricity \( \epsilon \)?

We received two correct solutions and present the solution submitted by the proposer.

Since \( m \neq n \), the perpendicular to \( m \) through \( F \) and the perpendicular \( t \) to \( n \) at \( A \) intersect, say at \( K \). Note that \( K \) is distinct from both \( F \) and \( A \) (since \( F \neq A \)). Define \( p \) to be the perpendicular to \( BF \) through \( K \). Then \( A \notin p \) (otherwise we would have \( t \perp BF \), implying \( n \parallel BF \), a contradiction). We also have \( F \notin p \) (otherwise \( KF \perp BF \), implying \( BF = m \) and \( B \in m \), contradicting \( B \neq A \)).

We first show uniqueness: Suppose that \( C_F \) exists. Note that \( t \) is the tangent to \( C_F \) at \( A \). Since \( \angle KFA = 90^\circ \) and \( K \in t \), \( K \) must be on the directrix of \( C_F \) associated with \( F \) (see [2], Theorem 1 p. 14). Thus, \( C_F \) must be the unique conic with focus \( F \), directrix \( p \) and eccentricity \( \frac{AF}{d(A,F)} \). Conversely, because the line \( p \) misses the distinct points \( A \) and \( F \), we can consider the unique conic with focus \( F \), directrix \( p \) and eccentricity \( \frac{AF}{d(A,F)} \). This conic passes through \( A \) (by the definition of eccentricity) and is tangent to \( AK \) at \( A \) (since \( K \in p \) and \( \angle KFA = 90^\circ \)); it therefore intersects \( n \) orthogonally at \( A \). Also, its focal axis is \( BF \) (since \( BF \perp p \)). Thus, this conic satisfies the required conditions for \( C_F \).

Note that the eccentricity of \( C_F \) is also equal to \( \frac{EB}{EA} \) (see [2], Theorem 4, p. 18) and that if \( F, F' \) are two distinct points on \( m \) (\( F, F' \neq A \)), then the conics \( C_F \) and \( C_{F'} \) intersect at a point other than \( A \)
are distinct (their focal axes are distinct). From these remarks, we see that there are as many conics $C_F$ with eccentricity $\epsilon$ as points of $M \in m$ that belong to the locus $E$ of points for which $\frac{MB}{MA} = \epsilon$. If $\epsilon = 1$, $E$ is the perpendicular bisector of $AB$; it intersects $m$ (since $m$ and $n$ are not perpendicular), so that exactly one conic $C_F$ is a parabola. If $\epsilon \neq 1$, then $E$ is a circle—the circle of Apollonius—which can intersect $m$ in at most two points. The collection of all these circles (as $\epsilon$ varies over the positive real numbers except 1) forms a nonintersecting pencil of circles with limiting points $A$ and $B$, one through each point of the plane not on the perpendicular bisector of $AB$ (see [1], Section 6.6). It follows that there are at most two conics $C_F$ corresponding to a given value of $\epsilon$.

To be more specific, $E$ has diameter $JJ'$ where $J, J'$ are the points of $n$ defined by $(1 + \epsilon)\overrightarrow{AJ} = \overrightarrow{AB}$ and $(1 - \epsilon)\overrightarrow{AJ'} = \overrightarrow{AB}$. The centre $U$ of $E$ is such that $(1 - \epsilon^2)\overrightarrow{AU} = \overrightarrow{AB}$; its radius is $\rho = JU = \epsilon AU = \frac{\epsilon AB}{1 - \epsilon^2}$. If $H, H'$ are the orthogonal projections of $B, U$ onto $m$, respectively, then $\frac{UH'}{BH} = \frac{AU}{AB} = \frac{1}{1 - \epsilon^2}$; hence $UH' = \frac{BH}{1 - \epsilon^2}$ (where $BH$ is the distance from $B$ to $m$ which, of course, is always less than $AB$). We conclude that no, one, or two conics $C_F$ have eccentricity $\epsilon$ according as $UH'$ is greater than, equal to, or less than $\rho$, which is equivalent to $\epsilon$ less than, equal to, or greater than $\frac{BH}{AB}$. So, for example, $C_F$ could never be a circle (for which $\epsilon = 0$).

References


4044. Proposed by Dragoljub Milošević.

Let $x, y, z$ be positive real numbers such that $x + y + z = 1$. Prove that

$$\frac{x+1}{x^3+1} + \frac{y+1}{y^3+1} + \frac{z+1}{z^3+1} \leq \frac{27}{7}.$$

We received 24 submissions, of which 22 were correct and complete. There were two main approaches: Jensen’s inequality, or comparing each term to a linear function. We present two solutions, one for each approach.

Solution 1, by Fernando Ballesta Yagüe.

As $x + y + z = 1$ and $x, y, z$ are positive, we have $x, y, z \in (0, 1)$. For $x$ in the interval $(0, 1)$, consider the rational function

$$f(x) = \frac{x+1}{x^3+1} = \frac{1}{x^2-x+1}.$$
Let’s take the second derivative to check its convexity:

\[
f'(x) = \frac{-2x + 1}{(x^2 - x + 1)^2},
\]

\[
f''(x) = \frac{-2(x^2 - x + 1)^2 - (-2x + 1) \cdot 2 \cdot (x^2 - x + 1) \cdot (2x - 1)}{(x^2 - x + 1)^4} = \frac{6x(x - 1)}{(x^2 - x + 1)^3}.
\]

Since \(x \in (0,1)\), we have \(x - 1 < 0\), but \(6x > 0\) and \(x^2 - x + 1 > 0\) (note that \(-x + 1 > 0\) for \(x \in (0,1)\)). So in the interval \(x \in (0,1)\) we have \(f''(x) < 0\) and hence \(f\) is concave. By Jensen’s Inequality for concave functions,

\[
\frac{1}{3}(f(x) + f(y) + f(z)) \leq f\left(\frac{1}{3}(x + y + z)\right) = f\left(\frac{1}{3}\right) = \frac{9}{7},
\]

in other words,

\[
\frac{1}{x^2 - x + 1} + \frac{1}{y^2 - y + 1} + \frac{1}{z^2 - z + 1} \leq \frac{27}{7},
\]

which is equivalent to the inequality we wanted to prove. Note that equality holds when \(x = y = z = \frac{1}{3}\).

**Solution 2, by Paul Bracken.**

As in the previous solution, define \(f(x) = \frac{1}{x^2 - x + 1}\) and show that \(f\) is concave for \(x \in (0,1)\). Therefore, if \(t(x)\) is a tangent line to \(f(x)\) at some point \(x_0 \in (0,1)\) then the inequality \(f(x) \leq t(x)\) holds for \(x \in (0,1)\). Let us calculate the tangent line to \(f(x)\) at \(x_0 = \frac{1}{3}\):

\[
t(x) = f'\left(\frac{1}{3}\right) \cdot \left(x - \frac{1}{3}\right) + f\left(\frac{1}{3}\right) = \frac{27}{49} x + \frac{54}{49}.
\]

The inequality \(f(x) \leq t(x)\) then gives us \(\frac{x + 1}{x^3 + 1} \leq \frac{27}{49} x + \frac{54}{49}\) for \(x \in (0,1)\). We obtain similar inequalities by replacing \(x\) by \(y\) and \(z\) respectively, then add the three inequalities to get

\[
\frac{x + 1}{x^3 + 1} + \frac{y + 1}{y^3 + 1} + \frac{z + 1}{z^3 + 1} \leq \frac{27}{49}(x + y + z) + 3 \cdot \frac{54}{49} = \frac{27}{7},
\]

where for the last equality we used \(x + y + z = 1\).

**4045. Proposed by Galav Kapoor.**

Suppose that we have a natural number \(n\) such that \(n \geq 10\). Show that by changing at most one digit of \(n\), we can compose a number of the form \(x^2 + y^2 + 10z^2\), where \(x, y, z\) are integers.

We received two correct solutions. We present the solution by Roy Barbara.

Recall that Legendre’s three-square theorem states that a natural number is a sum of three squares if and only if it is not of the form \(4^m(8k + 7)\). In particular, any natural number of the form \(4k + 2\) is a sum of three squares.
Now let $2k+1$ be any odd natural number. Then we can write $4k+2 = a^2+b^2+c^2$. Using $a^2+b^2+c^2 \equiv 2 \pmod{4}$, it is clear that exactly one of $a, b, c$ is even, say $c$. Setting $x = \frac{1}{2}(a+b), y = \frac{1}{2}(a-b), c = 2z$ yields

$$4k + 2 = 2x^2 + 2y^2 + 4z^2,$$

whence

$$2k + 1 = x^2 + y^2 + 2z^2.$$

Finally let $n \geq 10$. By changing the last digit of $n$ to a 5 (if necessary), we obtain a number of the form $10k + 5$ for which we have

$$10k + 5 = 5x^2 + 5y^2 + 10z^2 = (2x + y)^2 + (2y - x)^2 + 10z^2.$$

4046. Proposed by Michel Bataille.

Let $a, b, c$ be nonnegative real numbers such that $\sqrt{a} + \sqrt{b} + \sqrt{c} \geq 1$. Prove that

$$a^2 + b^2 + c^2 + 7(ab + bc + ca) \geq \sqrt{8(a+b)(b+c)(c+a)}.$$

Two correct solutions were received. A purported counterexample that was submitted had an error. We present both solutions.

Solution 1, by Madhav R. Modak.

$$\sqrt{8(a+b)(b+c)(c+a)}$$

$$\leq (\sqrt{a} + \sqrt{b} + \sqrt{c})\sqrt{8(a+b)(b+c)(c+a)}$$

$$= \sqrt{(4ab + 4ca)[2(a+b)(c+a)] + \sqrt{(4bc + 4ab)[2(a+b)(b+c)]}}$$

$$+ \sqrt{(4ca + 4bc)[2(b+c)(c+a)]}$$

$$\leq \frac{1}{2}[(4ab + 4ca) + 2(a+b)(c+a)] + \frac{1}{2}[(4bc + 4ab) + 2(a+b)(b+c)]$$

$$+ \frac{1}{2}[(4ca + 4bc) + 2(b+c)(c+a)]$$

$$= 4(ab + bc + ca) + (a^2 + ab + ca + bc) + (b^2 + ab + bc + ca) + (c^2 + ca + bc + ab)$$

$$= a^2 + b^2 + c^2 + 7(ab + bc + ca),$$

which yields the desired result.

Solution 2, by the proposer.

Since

$$(a^2 + 3ab + 3ca + bc)^2 = 8a(a+b)(b+c)(c+a) + (a-b)^2(a-c)^2,$$
it follows that
\[ a^2 + 3ab + 3ca + bc \geq \sqrt{a}(\sqrt{8(a + b)(b + c)(c + a)}). \]

Similarly
\[ b^2 + 3ab + 3bc + ca \geq \sqrt{b}(\sqrt{8(a + b)(b + c)(c + a)}) \]
and
\[ c^2 + 3ca + 3bc + ab \geq \sqrt{c}(\sqrt{8(a + b)(b + c)(c + a)}). \]

Adding these three inequalities yields the result.

*Editor’s comment.* Equality holds if and only if \( a = b = c = \frac{1}{9} \).

**4047. Proposed by Abdilkadir Altıntaş.**

Let \( ABC \) be a triangle with circumcircle \( O \), orthocenter \( H \) and \( \angle BAC = 60^\circ \). Suppose the circle with centre \( Q \) is tangent to \( BH \), \( CH \) and the circumcircle of \( ABC \). Show that \( OH \perp HQ \).

*All 14 submissions we received were correct. We feature two solutions.*
Solution 1 is a composite of solutions by Václav Konečný and Edmund Swylan.

The statement of the problem is faulty: Because the plane is partitioned into as many as eight regions by the circumcircle of triangle ABC and the lines HB and HC, there could be eight tritangent circles and, consequently, eight choices for Q, of which some lie on the Euler line OH (in which case the lines OH and HQ would be coincident, not perpendicular).

[Editor’s comment: Since the centres of all circles tangent to HB and HC would lie on a bisector of ∠BHC, the requirement that the circle be tangent also to the circumcircle was perhaps included to limit the choice of tritangent circle to the incircle of the curvilinear triangle HBC (formed by the line segments HB and HC and the circular arc BC). Then the problem has been correctly stated for an acute ∆ABC, but it is still not correct when there is an obtuse angle at B or C.]

The exact location of Q is not relevant to the correct theorem:

For any circle tangent to the lines HB and HC, its centre Q must belong to one of the two bisectors of ∠BHC, and so must O.

The claim for Q is a familiar theorem, while the claim for O depends on ∠BAC = 60° and must be proved.

As in the figure, denote the midpoints of AC and AB by M and N, respectively, and the feet of the altitudes to these lines by F and G. Then the segments FM and GN are congruent:

\[ FM = |FA - MA| = \left| AB \cdot \cos 60^\circ - \frac{AC}{2} \right| = \frac{1}{2}|AB - AC| \]

and

\[ GN = |GA - NA| = \left| AC \cdot \cos 60^\circ - \frac{AB}{2} \right| = \frac{1}{2}|AC - AB|. \]

Then the lines BF, CG, MO, NO form the sides of a rhombus for which the line OH is a diagonal. Thus OH bisects one of the angles formed by the lines HF and HG, as claimed.

Solution 2 is a composite of similar solutions by Šefket Arslanagić, Ricardo Barroso Campos, Prithwijit De (done independently), and Adnan Ibrić with Salem Malikić.

As in the figure that accompanies the statement of the problem, we assume that the given triangle is acute, and that F and G are the feet of the altitudes from B and C, respectively. Observe that ∠BHC = ∠FHG = 120° (since ∠A is 60° and is opposite ∠FHG in the circle whose diameter is AH). Furthermore, ∠BOC = 120° also (because O is the centre of the circumcircle so that the angle there is twice the angle BAC = 60° which is inscribed in that circle). Because O and H are on the same side of BC, it follows that B, O, H, C are concyclic. Finally, note that because ∆BOC is isosceles, ∠OCB = 30°. Since HQ is the
bisector of $\angle BHC$,$$
\angle BHQ = \frac{1}{2} \angle BHC = 60^\circ.
$$
Combine this with
$$
\angle OHB = \angle OCB = 30^\circ,
$$
and conclude that
$$
\angle OHQ = \angle OHB + \angle BHQ = 90^\circ.
$$

Editor’s Comments. Essentially the same problem has appeared before in *Crux* [1988: 165; 1990: 103] as Problem M1046, which was taken from the 1987 U.S.S.R journal *Kvant*:

If $\angle A = 60^\circ$ then one of the bisectors of the angle between the altitudes from $B$ and $C$ passes through $O$.

This and related properties were discussed under the heading “Property 3” in the article “Recurring *Crux* Configurations 3: Triangles Whose Angles Satisfy $2B = C + A$” [2011: 350].

4048. Proposed by Leonard Giugiuc and Daniel Sitaru.

Let $n \geq 2$ be an integer and let $a_k \geq 1$ be real numbers, $1 \leq k \leq n$. Prove the inequality
$$
a_1a_2 \cdots a_n - \frac{1}{a_1a_2 \cdots a_n} \geq \left( a_1 - \frac{1}{a_1} \right) + \left( a_2 - \frac{1}{a_2} \right) + \cdots + \left( a_n - \frac{1}{a_n} \right)
$$
and study equality cases.

Thirteen solutions were received, all of which established the inequality. Two of them did not get all the possible conditions for equality, while three others neglected to consider when equality occurred. The solutions were all similar to the one presented below.

Let
$$
f(x) = x - \frac{1}{x}
$$
and observe that, for $x, y \geq 1$,
$$
f(xy) - f(x) - f(y) = (xy)^{-1}(xy - 1)(x - 1)(y - 1) \geq 0
$$
with equality if and only if at least one of $x$ and $y$ is equal to 1.

We establish the result by induction.

The foregoing shows that it is true for $n = 2$. Suppose that the inequality holds for $n = m \geq 2$ with equality iff all but at most one of $a_1, a_2, \ldots, a_m$ is equal to 1. Then, by the foregoing property of $f$ and the result for $n = m$,
$$
f(a_1a_2 \cdots a_m a_{m+1}) \geq f(a_1a_2 \cdots a_m) + f(a_{m+1}) \geq \sum_{k=1}^{m} f(a_k) + f(a_{m+1}) = \sum_{k=1}^{m+1} f(a_k).
$$
Equality holds if and only if either
\[ a_1 a_2 \cdots a_m = 1, \] in which case \( a_1 = a_2 = \cdots = a_m = 1, \) or
\[ a_{m+1} = 1 \] and 
\[ f(a_1 a_2 \cdots a_m) = \sum_{k=1}^{m} f(a_k). \]
In either case, all but at most one of \( a_1, a_2, \ldots, a_{m+1} \) is equal to 1.

Editor’s comments. One can also peel off the last two terms in the product so that the induction step becomes
\[ f(a_1 a_2 \cdots a_{m+1}) \geq \sum_{k=1}^{m-1} f(a_k) + f(a_m a_{m+1}) \geq \sum_{k=1}^{m+1} f(a_k). \]

Edmund Swyland observed that if, for any \( i \) and \( j, \) you replaced the pair \((a_i, a_j)\) by \((a_i a_j, 1), \) the left side \( f(a_1 a_2 \cdots a_n) \) of the inequality remained unchanged, but the right side increased. Thus we can reduce the problem to establishing that it holds when all but two of the \( a_i \) are equal to 1, and this now involves dealing with the case \( n = 2. \)

Kee-Wai Lau pointed out that an easy induction argument yields
\[ f(a_1 a_2 \cdots a_n) - \sum_{k=1}^{n} f(a_k) = \sum_{k=2}^{n} \frac{(a_1 a_2 \cdots a_{k-1} - 1)(a_k - 1)(a_1 a_2 \cdots a_k - 1)}{a_1 a_2 \cdots a_k}. \]

4049. Proposed by Mihaela Berindeanu.

Evaluate
\[ \int \frac{\sin x - x \cos x}{(x + \sin x)(x + 2 \sin x)} \, dx \]
for all \( x \in (0, \pi/2). \)

We received 16 submissions all of which were correct. We present a composite of the nearly identical solutions given by Adnan Ali, Michel Bataille, Prithwijit De, Joseph Ling and Albert Stadler, all done independently

Let \( I \) denote the given integral. Since it is readily checked that
\[ (1 + \cos x)(x + 2 \sin x) - (1 + 2 \cos x)(x + \sin x) = \sin x - x \cos x, \]
we have
\[ I = \int \left( \frac{1 + \cos x}{x + \sin x} - \frac{1 + 2 \cos x}{x + 2 \sin x} \right) \, dx \]
\[ = \ln(x + \sin x) - \ln(x + 2 \sin x) + C \]
\[ = \ln \left( \frac{x + \sin x}{x + 2 \sin x} \right) + C, \]
where \( C \) is an arbitrary constant.

Crux Mathematicorum, Vol. 42(5), May 2016
4050. Proposed by Mehtaab Sawhney.

Prove that

$$\sum_{k=0}^{2n} \frac{4n}{k, k, 2n-k, 2n-k} = \left(\frac{4n}{2n}\right)^2$$

for all nonnegative integers $n$.

We received twelve correct solutions which were split between an arithmetic proof and a proof by double counting, so we present a solution of each type.

Solution 1, by C.R. Pranesachar.

We have

$$\sum_{k=0}^{2n} \frac{4n}{k, k, 2n-k, 2n-k} = \sum_{k=0}^{2n} \frac{4n}{2n} \binom{2n}{k}^2$$

$$= \left(\frac{4n}{2n}\right)^2 \sum_{k=0}^{2n} \frac{4n}{2n} \frac{\binom{2n}{k}}{\binom{2n}{2n-k}}$$

$$= \left(\frac{4n}{2n}\right)^2 \binom{2n}{n},$$

where the last equality is due to Vandermonde’s identity.

Solution 2, by Joseph DiMuro.

Let’s say we have a classroom with $4n$ students. The teacher wants to choose $2n$ of them to work on one project and $2n$ of them to work on a second project (independently of each other – students may be assigned to both or neither of the projects). In how many ways can the teacher assign students to the projects?

On one hand there are $\binom{4n}{2n}$ ways to choose the students for each of the two projects, thus $\left(\binom{4n}{2n}\right)^2$ possibilities altogether.

On the other hand note that if $k$ students are assigned to both projects then $2n-k$ will be assigned to just the first project, $2n-k$ to just the second project, and $k$ to neither project. So the teacher can proceed as follows: first decide on the number $k$ of students that will be assigned to both projects, then partition the class into four groups – of size $k$ (both projects), $2n-k$ (first project), $2n-k$ (second project), and $k$ (neither project). There are

$$\sum_{k=0}^{2n} \frac{4n}{k, k, 2n-k, 2n-k}$$

ways to do this. Thus the two sides in the problem are equal.